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Stability of post-fertilization traveling waves

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ABSTRACT

This paper studies the stability of a family of traveling wave solutions to the system proposed by Lane et al. [D.C. Lane, J.D. Murray, V.S. Manoranjan, Analysis of wave phenomena in a morphogenetic mechanochemical model and an application to post-fertilization waves on eggs, IMA J. Math. Appl. Med. Biol. 4 (4) (1987) 309–331], to model a pair of mechanochemical phenomena known as post-fertilization waves on eggs. The waves consist of an elastic deformation pulse on the egg's surface, and a free calcium concentration front. The family is indexed by a coupling parameter measuring contraction stress effects on the calcium concentration. This work establishes the spectral, linear and nonlinear orbital stability of these post-fertilization waves for small values of the coupling parameter. The usual methods for the spectral and evolution equations cannot be applied because of the presence of mixed partial derivatives in the elastic equation. Nonetheless, exponential decay of the directly constructed semigroup on the complement of the zero eigenspace is established. We show that small perturbations of the waves yield solutions to the nonlinear equations decaying exponentially to a phase-modulated traveling wave.

1. Introduction

Post-fertilization traveling waves are pairs of mechanical and chemical phenomena observed on the surface of some vertebrate eggs shortly after fertilization. These waves consist of a transition front of the free calcium concentration, and a contraction elastic wave on the egg's surface. These phenomena have been observed experimentally under various circumstances [21,3,40]. In an attempt to model

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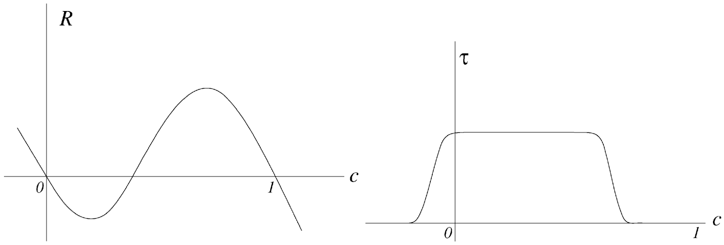

Fig. 1. Typical forms of the nonlinear functions $R(c)$ (left) and $\tau(c)$ (right). Notice that $R(c)$ has a bistable shape, with $R'(0) < 0$ and $R'(1) < 0$. Phenomenological considerations [24] prescribe the contractile forces τ to switch from $\tau = 1$ at $c = 0$, to $\tau = 0$ at $c = 1$ (as shown); for the purposes of the present study it suffices to assume τ of class C^2 with compact support in $\mathbb{R}$ (and consequently, $\tau'(c)$ is bounded).

the occurrence and behavior of these waves, Lane et al. [24] proposed a mechanochemical system of equations which underlies solutions of wave-type. Based on the cytoskeleton model of Murray and Oster [27] (which assigns elastic properties to cytoplasmic material), the proposed system combines the effects of the free calcium of the egg's surface on the elastic stress, and the effects of stretching on the calcium concentration itself. In order to analyze wave-like solutions, Lane et al. [24] considered one component of the deformation variable, leading to the following system of equations:

$$\begin{aligned} \mu u_{xxt} + u_{xx} - \tau(c)_x - su &= 0, \\ c_t - Dc_{xx} - R(c) - \epsilon u_x &= 0, \end{aligned} \tag{1}$$

with $(x,t) \in \mathbb{R} \times [0,+\infty)$ denoting (one-dimensional) space and time. In (1), u denotes the elastic deformation on the egg's surface, and c is the concentration of free calcium. The nonlinear terms $\tau(c)$ and $R(c)$ represent contractile forces acting on the egg's surface and autocatalytic effects, respectively. In addition, $\mu = \mu_1 + \mu_2 > 0$ measures the combined shear and bulk viscosities; the parameter $1 \gg s > 0$ is the restoring force of the egg's surface; $D > 0$ represents the Fick's diffusion constant of calcium; and $\epsilon > 0$ accounts for the contraction stress effects on the increase of c (see [16,24] for details).

The nonlinear functions R and τ are assumed to be smooth, and R has a bistable shape; more precisely we suppose that

$$R(c) \text{ and } \tau(c) \text{ are functions of class } C^2, \tag{2}$$

$$\tau(c) \text{ has compact support in } \mathbb{R} \ \big(\text{i.e. } \tau \in C_0^2(\mathbb{R})\big), \tag{3}$$

$$R(0) = R(1) = R(c_0) = 0, \quad \text{for some } c_0 \in (0,1), \tag{4}$$

$$\int_0^1 R(c)\,dc > 0, \quad \text{and} \tag{5}$$

$$R'(0) < 0, \qquad R'(1) < 0. \tag{6}$$

Typical nonlinear functions for (1) are depicted in Fig. 1. Further discussion on the model can be found in [24] and in Section 7 below.

Consider a traveling wave solution to (1), of form $(\bar{u},\bar{c})(x+\theta t)$, where $\theta \in \mathbb{R}$ denotes the speed of the wave. Make the Galilean transformation

$$x \to x + \theta t,$$

and denote $\partial/\partial x$ (or subscript $(\cdot)_x$) as differentiation with respect to the moving variable. Hence $(\bar{u},\bar{c})(x)$ satisfies the following system of equations

$$\begin{aligned} &\mu\theta\bar{u}_{xxx} + \bar{u}_{xx} - \tau(\bar{c})_x - s\bar{u} = 0,\\ &\theta\bar{c}_x - D\bar{c}_{xx} - R(\bar{c}) - \epsilon\bar{u}_x = 0. \end{aligned} \tag{7}$$

We are interested in bounded solutions $(\bar{u},\bar{c})$ to (7) with $\bar{u}(\pm\infty) = 0$ and $\bar{c}(+\infty) = 1, \bar{c}(-\infty) = 0$; in other words, $\bar{u}$ is an elastic pulse and $\bar{c}$ is a calcium concentration transition front. The pair $(\bar{u},\bar{c})$ corresponds to a heteroclinic orbit of system (7). As described in [16], the restoring force of the egg's surface $s > 0$ is crucial for the existence of the elastic pulse (otherwise absent).

In [16], the first author and collaborators proved the existence of traveling wave solutions to (7) for small values of $0 \leqslant \epsilon \ll 1$. First, they showed that there exists a unique heteroclinic connection for the uncoupled system with $\epsilon = 0$,

$$\begin{aligned} &\mu\theta\bar{u}_{xxx} + \bar{u}_{xx} - \tau(\bar{c})_x - s\bar{u} = 0,\\ &\theta\bar{c}_x - D\bar{c}_{xx} - R(\bar{c}) = 0, \end{aligned} \tag{8}$$

which consists of a previously well-studied bistable front [14] (also known as the *Nagumo front* [4, 22,28]), and an elastic pulse which depends, in turn, on the calcium wave. Thereafter, they proved that this heteroclinic orbit persists for small values of the coupling parameter $\epsilon > 0$, using Melnikov's integral method [19,20], and thus showing the existence of wave solutions $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ to (7). Moreover, the speed of propagation θ is uniquely determined for each value of $\epsilon \geqslant 0$.

The present paper addresses the stability under small perturbations of these post-fertilization traveling waves, in the regime of small values of the coupling parameter $\epsilon \geqslant 0$. Motivated by the modulation theory of Whitham [41] (see also [33,26]), we decompose the solution to (1) into a modulating traveling wave with phase depending on time, plus a perturbation, having the form

$$(\tilde{u},\tilde{c})(x,t) = (u,c)\big(x+\theta t+\alpha(t),t\big) + \big(\bar{u}^\epsilon,\bar{c}^\epsilon\big)\big(x+\theta t+\alpha(t)\big), \tag{9}$$

where $\alpha(t)$ is the phase and (u,c) stands for the perturbation. The initial condition for the whole solution $(\tilde{u},\tilde{c})$ is a small perturbation (u_0,c_0) plus a translated traveling wave with constant phase α_0. A suitable phase function $\alpha(t)$ is constructed such that the perturbation (u,c) decays exponentially in time in a suitable norm and the wave experiences a slight shift in phase. This property is called *orbital asymptotic stability* in the literature.

In order to achieve this goal, we present our stability analysis in a three step program. The first part establishes *spectral stability*, yielding the conditions under which solutions to the resulting linearized system of equations are "well-behaved", precluding spatially bounded solutions with explosive behavior in time of form $(u,c)(x)e^{\lambda t}$, $\operatorname{Re}\lambda > 0$. The second step consists on the construction and the study of decaying properties of the semigroup associated to the solutions of the linearized perturbation equations around the waves. The third and final part pertains to the orbital asymptotic stability of the solutions to the nonlinear equations.

At this point we observe, however, that the system has mixed derivatives in space and time in the elastic equation and that the resulting spectral and evolution systems for the perturbations are written in non-standard form. Therefore, the usual methods to analyze the spectrum of a linearized operator and its semigroup-generating properties cannot be applied. This feature reflects itself on both the dynamics and the behavior of the solutions.

To circumvent this difficulty, we make the observation that the reformulation of the spectral problem of Alexander, Gardner and Jones [1] (which recasts the equations as a first order system with the eigenvalue as a parameter) does not depend on the explicit form of the linearized operator around the wave. This property allows us to make precise a definition of resolvent and spectra suitable for our needs (see, e.g., Sandstede [36] and Definition 2.6 below). This definition is provided in terms of the Fredholm properties of the first order operators and it coincides with the usual definition of spectra and resolvent of a linearized operator around the wave [36,37] in the standard case. As far as we know, our analysis is the first that applies the flexibility of the formulation of [1] to a problem in non-standard form. The Evans function (first introduced by J.W. Evans in the study of nerve axon waves [10–13], and reformulated in a more general setting by Alexander et al. [1]) is a powerful tool to locate isolated eigenvalues in the point spectrum and near the essential spectrum. We use Evans function tools to prove that traveling wave solutions to (1) for $\epsilon > 0$ sufficiently small are spectrally stable, that is, the spectrum of the linearized system around the wave is located in the stable complex half plane $\{\mathrm{Re}\,\lambda < 0\} \cup \{0\}$. For that purpose, we apply a result from Evans function theory, developed in the context of viscous shocks [34], which assures that, under suitable structural but rather general conditions, the Evans functions for $\epsilon > 0$ converge uniformly to that of the uncoupled traveling wave with $\epsilon = 0$. By analyticity and uniform convergence, the non-vanishing property of the Evans function for $\epsilon = 0$ persists for $\epsilon > 0$ sufficiently small. Moreover, we show that the zero eigenvalue associated to translation invariance of the waves persists with same algebraic multiplicity equal to one, and that there is a uniform spectral gap between the essential spectrum and the imaginary axis. A similar convergence approach in a singular limit was implemented to prove the existence of an unstable eigenvalue for the slow pulse of the FitzHugh–Nagumo system in [15].

These spectral bounds are crucial for the establishment of the decaying properties for the linearized system. Once again, since the linearized equations are not in standard form, the usual circumstances for the application of the results of generation of semigroups cannot be easily verified. We show, however, that the solutions to the linearized perturbation equations generate a C_0-semigroup by means of a direct construction. For this purpose, we express the system in terms of the deformation gradient variable u_x, and the constructed C_0-semigroup acts on the Sobolev space $H^1 \times H^1$ in the variables (u_x, c). The solutions decay exponentially in the H^1 norm on the spectral complement of its zero eigenspace. To prove this decay behavior we apply the Gearhart–Prüss criterion [18,35] of C_0-semigroups in Hilbert spaces.

Finally, we employ the exponential decay rates of the semigroup to show nonlinear asymptotic stability. We follow the method of Pego and Weinstein [33] in the study of solitary waves for KdV-type equations. We remark that our analysis is considerably simpler, due to the fact that the manifold generated by the waves in the present situation is a one-dimensional family, with the phase as a parameter (the wave speed is uniquely determined). In order words, the Jordan block associated to the zero eigenvalue has size equal to one (the isolated eigenvalue is simple), in contrast with the situation in [33]. In this fashion, we construct a suitable nonlinear ansatz (9) for which the phase modulates and the perturbation has the desired exponential decay.

Plan of the paper. This work is structured as follows. In Section 2 we obtain the perturbation and spectral equations, and provide a definition of resolvent, spectra and spectral stability; we recall the main existence results of [16] and the basic properties of the traveling waves; and we pose the perturbation equations in terms of the new variables. Section 3 contains the statements of the main results of the paper. Section 4 is devoted to prove spectral stability (see Theorem 1 below). We show that our spectral problem satisfies the main assumptions for the application of the Evans function machinery and prove that the coupled spectral problem with $\epsilon > 0$ satisfies the conditions for uniform convergence of the associated family of Evans functions. The central Section 5 contains both the construction of the C_0-semigroup generated by the linearized equations and the proof of the exponential decay properties (Theorem 2 below). Section 6 provides the detailed nonlinear asymptotic analysis. We justify the representation of the solutions to the nonlinear system as phase-modulated waves and perturbations, and provide exponential decay rates for the latter using the semigroup estimates obtained before. We include Appendix A containing a global existence result for the nonlinear system written in terms of the deformation gradient. Finally, in Section 7, we outline extensions to multidimensions, and make some general remarks about the present study, both at the theoretical and phenomenological level.

Notation. In the sequel, $*$ denotes complex conjugation for scalars, vectors or matrices, whereas $^\top$ denotes simple transposition. The spaces $W^{m,p}$, H^m, $m \geqslant 0$, $1 \leqslant p < +\infty$, with $L^2 = H^0$, will denote the standard scalar Sobolev spaces in $\mathbb{R}$, except where it is otherwise explicitly stated. The symbol "$\lesssim$" means "$\leqslant$" times a harmless positive constant.

2. Formulation of the spectral and perturbation equations

2.1. Structure of traveling waves

Let us recall the existence results of [16,14] which will be needed later.

Proposition 2.1. *(See [14,2,5,16].) Under assumptions* (2)–(6)*, Eqs.* (8) *have a unique solution* $(\bar{u}, \bar{c})(x)$ *satisfying* $\bar{u}(\pm\infty) = 0$, $\bar{c}(+\infty) = 1$, $\bar{c}(-\infty) = 0$, *where the wave speed* $\theta = \theta_0$ *is determined uniquely by*

$$\theta_0 := \frac{\int_0^1 R(c)\,dc}{\int_{\mathbb{R}} \bar{c}_x(x)^2\,dx} > 0. \tag{10}$$

Moreover, $\bar{c}$ *is strictly monotone,* $\bar{c}_x > 0$. *Here* $s > 0$ *must satisfy* $0 < \mu\sqrt{s} < 2\theta_0/\sqrt{27}$.

Remark 2.2. Details on the existence of the calcium front, the uniqueness of θ_0, and the global monotonicity of $\bar{c}$ can be found in [14] (see also [2,5]). θ_0 being independent of s, we may choose s in the region $0 < \mu\sqrt{s} < 2\theta_0/\sqrt{27}$ to preclude waves oscillating around $c \equiv 1$ (see [16] for details). We keep this assumption for the rest of the paper.

Proposition 2.3. *(See [16].) Assume* $\epsilon > 0$ *is sufficiently small. Then system* (1) *admits a traveling wave solution* $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ *satisfying* (7) *and* $\bar{u}^\epsilon(\pm\infty) = 0$, $\bar{c}^\epsilon(+\infty) = 1$, $\bar{c}^\epsilon(-\infty) = 0$. *The wave speed is given by* $\theta(\epsilon) = \theta_0 + o(1)$ *for* $\epsilon \sim 0^+$.

As a by-product of the existence theorems we have uniform exponential decay of the traveling waves.

Lemma 2.4. *For all* $\epsilon \geqslant 0$ *sufficiently small, traveling wave solutions* $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ *satisfy*

$$\begin{aligned}
\left|\partial_x^j \bar{u}^\epsilon(x)\right| &\lesssim e^{-|x|/C_1}, \quad \text{as } |x| \to +\infty,\ j = 0,1,2,3,\\
\left|\partial_x^i\big(\bar{c}^\epsilon(x) - 1\big)\right| &\lesssim e^{-x/C_1}, \quad \text{as } x \to +\infty,\ i = 0,1,2,\\
\left|\partial_x^i \bar{c}^\epsilon(x)\right| &\lesssim e^{+x/C_1}, \quad \text{as } x \to -\infty,\ i = 0,1,2,
\end{aligned} \tag{11}$$

with some uniform $C_1 > 0$.

Proof. This is a direct consequence of hyperbolicity of the non-degenerate equilibrium points. The traveling wave pair $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ is a heteroclinic connection between the hyperbolic points $P_0 = 0$ and $P_1 = (0,0,0,1,0)^\top$, when Eqs. (7) are written as a first order system for $(u, u_x, u_{xx}, c, c_x)^\top$ (see [16]). The linearization of the system around P_n, $n = 0,1$, is given by $\mathbb{A}^\epsilon_\pm(0)$ (see (15)–(17) below), and has three roots with $\operatorname{Re}\kappa^\epsilon_j < 0$, and two roots with $\operatorname{Re}\kappa^\epsilon_j > 0$ for all $\epsilon \geqslant 0$ (see Theorem 1 in [16]). Whence, the dimensions of the stable and unstable spaces of $\mathbb{A}^\epsilon_\pm(0)$ are $\dim S^\epsilon_\pm(0) = 3$ and $\dim U^\epsilon_\pm(0) = 2$, respectively, and there is no center eigenspace. Since $|\operatorname{Re}\sigma(\mathbb{A}^0_\pm(0))| \geqslant \delta_0$ for some fixed $\delta_0 > 0$, by continuity on ϵ of the coefficients $\mathbb{A}^\epsilon_\pm(0)$ there exists $\epsilon_0 > 0$ such that there holds the uniform bound

$$\left|\operatorname{Re}\sigma\left(\mathbb{A}^\epsilon_\pm(0)\right)\right| \geqslant 1/C_1 > 0,$$

on $\epsilon \in [0, \epsilon_0]$, with $1/C_1 = \delta_0/2$. Exponential decay (11) follows by standard ODE estimates. $\square$

System of equations (7) is autonomous in the direction of propagation, that is, it is invariant under the transformation $x \to x + \alpha$, with $\alpha \in \mathbb{R}$; hence, every translate of the traveling waves, namely $(\bar{u}, \bar{c})(\cdot + \alpha)$, is also a connecting orbit and a solution to (7). Since the speed of propagation θ^ϵ is uniquely determined for each $\epsilon \geqslant 0$, the phase α is the only free parameter, and this yields a one-dimensional smooth manifold of solutions parametrized by $\alpha \in \mathbb{R}$. This property is called translation invariance of the traveling waves.

2.2. Perturbation equations and the spectral problem

Due to translation invariance, under which a small perturbation of a traveling wave can yield another wave with a permanent phase shift, the appropriate notion of stability is that of *orbital stability*, understood as the property that any solution, which at time $t = 0$ is sufficiently close to a translate of the wave, converges to another member of the family of translates as $t \to +\infty$. More precisely, we have the following

Definition 2.5. Let X and Y be Banach spaces. We say that the traveling wave solution $(\bar{u}, \bar{c})$ to (7) is *nonlinearly orbitally stable* if there exists $\delta > 0$ such that any solution (u, c) to (1) satisfying $\|(u, c)(\cdot, 0) - (\bar{u}, \bar{c})(\cdot)\|_X \leqslant \delta$ converges to a translate of the wave in Y, that is,

$$\big\|(u, c)(\cdot, t) - (\bar{u}, \bar{c})(\cdot + \alpha)\big\|_Y \to 0,$$

as $t \to +\infty$ for some $\alpha \in \mathbb{R}$. We refer to the latter property as *nonlinear orbital $X \to Y$ stability*.

Let us consider solutions to (1) of form $u + \bar{u}, c + \bar{c}$, u and c being perturbations. Making again the change of variables $x \to x + \theta t$ we obtain, in view of the profile equations (7), the following nonlinear system for u and c,

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - \big(\tau(c + \bar{c}) - \tau(\bar{c})\big)_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - \big(R(c + \bar{c}) - R(\bar{c})\big) &= 0. \end{aligned} \tag{12}$$

Linearizing (12) around the wave, we obtain the system

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - \big(\tau'(\bar{c})c\big)_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c &= 0. \end{aligned} \tag{13}$$

In order to define a suitable spectral problem, specialize (13) to perturbations of form $(e^{\lambda t}u(x), e^{\lambda t}c(x))$ with $\lambda \in \mathbb{C}$ to obtain the following linear system of equations parametrized by λ

$$\begin{aligned} \mu\theta u_{xxx} + (\mu\lambda + 1)u_{xx} - su - \big(\tau'(\bar{c})c\big)_x &= 0, \\ \lambda c + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c &= 0. \end{aligned} \tag{14}$$

It is clear that a necessary condition for stability of the traveling waves is the absence of non-trivial L^2 solutions $(u, c)(x)$ to system (14) with $\operatorname{Re}\lambda > 0$. Note that system (13) has not the usual form $U_t = \mathcal{L}U$, with $\mathcal{L}$ a linear operator, for which the spectral condition is simply that the unstable half plane $\{\operatorname{Re}\lambda \geqslant 0\}$ is contained in the resolvent set of $\mathcal{L}$. Therefore, we must make precise the notion of spectral stability suitable for our needs.

One can write system (14) as a first order ODE system in the frequency regime (according to custom in the Evans function literature [1,36]), of the form

$$W_x = \mathbb{A}^\epsilon(x, \lambda)W, \tag{15}$$

with

$$W := (u, u_x, u_{xx}, c, c_x)^\top,$$

$$\mathbb{A}^\epsilon(x,\lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta & \tau''(\bar{c})\bar{c}_x/\mu\theta & \tau'(\bar{c})/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(\bar{c}))/D & \theta/D \end{pmatrix}, \tag{16}$$

where $\theta = \theta(\epsilon)$ and $\bar{c} = \bar{c}^\epsilon$ are the speed and calcium wave from Proposition 2.3. By standard considerations [22], the asymptotic coefficient matrices at $x = \pm\infty$, namely

$$\mathbb{A}^\epsilon_\pm(\lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta & 0 & \tau'(n)/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(n))/D & \theta/D \end{pmatrix} \tag{17}$$

(where $n = 0, 1$, for $x = -\infty, +\infty$, respectively), determine the location of the essential spectrum. In fact, the stability of the essential spectrum is a consequence of the hyperbolicity of the asymptotic matrices in a connected region containing $\{\operatorname{Re}\lambda \geqslant 0\}$ (see Corollary 4.8 below).

Consider the following family of linear, closed, densely defined operators in $L^2(\mathbb{R}; \mathbb{C}^5)$,

$$\begin{aligned} \mathcal{T}^\epsilon(\lambda) : \mathcal{D}\big(\mathcal{T}^\epsilon\big) &\to L^2\big(\mathbb{R}; \mathbb{C}^5\big), \\ W &\mapsto W_x - \mathbb{A}^\epsilon(x,\lambda)W, \end{aligned} \tag{18}$$

with domain $\mathcal{D}(\mathcal{T}^\epsilon) = H^1(\mathbb{R}; \mathbb{C}^5)$, indexed by $\epsilon \geqslant 0$ and $\lambda \in \mathbb{C}$.

Definition 2.6. For fixed $\epsilon \geqslant 0$ we define the *resolvent* ρ, the *point spectrum* σ_{pt} and the *essential spectrum* σ_{ess} of problem (14) as

$$\rho := \big\{\lambda \in \mathbb{C}\colon\ \mathcal{T}^\epsilon(\lambda) \text{ is one-to-one and onto, and } \mathcal{T}^\epsilon(\lambda)^{-1} \text{ is bounded}\big\},$$

$$\sigma_{\mathrm{pt}} := \big\{\lambda \in \mathbb{C}\colon\ \mathcal{T}^\epsilon(\lambda) \text{ is Fredholm with index 0 and has a non-trivial kernel}\big\},$$

$$\sigma_{\mathrm{ess}} := \big\{\lambda \in \mathbb{C}\colon\ \mathcal{T}^\epsilon(\lambda) \text{ is either not Fredholm or has index different from 0}\big\}.$$

The *spectrum* σ of (14) is the union of the point and essential spectrum, $\sigma = \sigma_{\mathrm{ess}} \cup \sigma_{\mathrm{pt}}$. Note that since each $\mathcal{T}^\epsilon(\lambda)$ is closed, then $\rho = \mathbb{C} \backslash \sigma$ (see Kato [23, p. 167]). If $\lambda \in \sigma_{\mathrm{pt}}$ then we say that λ is an *eigenvalue* of (14).

We notice that the coefficients (16) are linear in λ and one can write

$$\mathbb{A}^\epsilon(x,\lambda) = \tilde{\mathbb{A}}^\epsilon_0(x) + \lambda\tilde{\mathbb{A}}^\epsilon_1(x), \quad x \in \mathbb{R}, \tag{19}$$

with

$$\tilde{\mathbb{A}}_0^\epsilon(x) := \mathbb{A}^\epsilon(x,0),$$

$$\tilde{\mathbb{A}}_1^\epsilon(x) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/D & 0 \end{pmatrix}. \tag{20}$$

Following Sandstede [36, Section 3.3] we define the multiplicity of an eigenvalue $\lambda \in \sigma_{\mathrm{pt}}$.

Definition 2.7. Assume $\lambda \in \sigma_{\mathrm{pt}}$. Its geometric multiplicity (*g.m.*) is the maximal number of linearly independent elements in $\ker \mathcal{T}^\epsilon(\lambda)$. Suppose $\lambda \in \sigma_{\mathrm{pt}}$ has $g.m. = 1$, so that $\ker \mathcal{T}^\epsilon(\lambda) = \mathrm{span}\{W_1\}$. We say λ has algebraic multiplicity (*a.m.*) equal to m if we can solve

$$\mathcal{T}^\epsilon(\lambda) W_j = \tilde{\mathbb{A}}_1^\epsilon(x) W_{j-1},$$

for each $j = 2, \dots, m$, with $W_j \in H^1$, but there is no H^1 solution W to

$$\mathcal{T}^\epsilon(\lambda) W = \tilde{\mathbb{A}}_1^\epsilon(x) W_m.$$

For an arbitrary eigenvalue $\lambda \in \sigma_{\mathrm{pt}}$ with $g.m. = l$, the algebraic multiplicity is defined as the sum of the multiplicities $\sum_k^l m_k$ of a maximal set of linearly independent elements in $\ker \mathcal{T}^\epsilon(\lambda) = \mathrm{span}\{W_1, \dots, W_l\}$.

Remark 2.8. There are different definitions of the essential spectrum (see, e.g., [8,23]). This choice allows us to apply Palmer's results [30,31] (which relate the Fredholm properties of the operators $\mathcal{T}^\epsilon$ with exponential dichotomies) directly to our problem. Note that this definition of spectra for (14) coincides with the definition of spectra of a linearized operator $\mathcal{L}$ in the case when the original equations are written as $U_t = \mathcal{L}U$, and the associated spectral problem, $\lambda U = \mathcal{L}U$, is recast as a first order system of the form $W_x = \mathbb{A}(x,\lambda)W$. This equivalence holds because the Fredholm properties of $\mathcal{L} - \lambda$ and $\mathcal{T}(\lambda)$ are the same. Moreover, the geometric and algebraic multiplicities, as well as the dimension of each Jordan block, are the same whether computed for $\mathcal{L} - \lambda$ or $\mathcal{T}(\lambda)$; see, for example, Sandstede [36], and Sandstede and Scheel [37,38].

By standard results [22,36,1], it is known that the stability of the essential spectrum is a consequence of the hyperbolicity of the asymptotic matrices $\mathbb{A}_\pm^\epsilon$ in a connected region containing $\{\mathrm{Re}\,\lambda \geqslant 0\}$ (see Corollary 4.8 below). Therefore, we define the spectral stability of the waves in terms of the point spectrum, meaning no loss of generality.

Definition 2.9. We define *strong spectral stability* of the traveling wave solutions of (7) as

$$\sigma_{\mathrm{pt}} \subset \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,\lambda < 0\} \cup \{0\}, \tag{21}$$

or equivalently [22], there are no L^2 solutions to the eigenvalue equations (14) for $\mathrm{Re}\,\lambda \geqslant 0$ and $\lambda \neq 0$.

2.3. The Evans function

If Ω is an open connected region in the complement of the essential spectrum, the Evans function [1,13,36] is an analytic function of $\lambda \in \Omega$ with the property that its zeroes coincide with isolated eigenvalues of finite multiplicity; furthermore, the order of the zero is the algebraic multiplicity (*a.m.*) of the eigenvalue. Eigenfunctions $W \in L^2$ of (14) are characterized by non-trivial intersection between the stable/unstable manifolds at $+\infty/-\infty$ of (15). The Evans function measures then the angle of this intersection. It is usually constructed by means of the Wronskian of ordered bases $W_1^-(x,\lambda)$, $W_2^-(x,\lambda)$ spanning the solutions to (15) that decay at $-\infty$, and $W_3^+(x,\lambda)$, $W_4^+(x,\lambda)$, $W_5^+(x,\lambda)$ spanning the solutions that decay at $+\infty$ (here the unstable manifold $U^-(\lambda)$ of $\mathbb{A}_-$ has dimension 2, and the stable manifold $S^+(\lambda)$ has dimension 3; see Lemma 4.7 below). The dimensions of these manifolds remain constant for $\lambda \in \Omega$, which is called the *region of consistent splitting* [1]. The Evans function is therefore given by

$$D^\epsilon(\lambda) = \det\big(W_1^-(x,\lambda), W_2^-(x,\lambda), W_3^+(x,\lambda), W_4^+(x,\lambda), W_5^+(x,\lambda)\big)_{|x=0}$$

(see [1,34,36,46]), measuring the transversality of the initial conditions that provide solutions decaying at both ends $x = \pm\infty$. The Evans function is analytic in the region of constant splitting Ω, and it has the following property: $D^\epsilon(\lambda) = 0$ if and only if there exists a non-trivial $W \in L^2$ solving (15), that is, it vanishes precisely at isolated eigenvalues. Moreover, the order of a root[1] λ_* of $D(\lambda)$ coincides with its algebraic multiplicity as an eigenvalue. The Evans function is highly non-unique, but they all differ by a non-vanishing analytic factor.

In addition, by translation invariance of the wave, $\lambda = 0$ is an eigenvalue with the derivative of the wave as eigenfunction, and thus $D^\epsilon(0) = 0$ if the essential spectrum does not touch zero. Proving spectral stability amounts to showing that the Evans function does not vanish in the closed right half plane except possibly for an isolated zero at $\lambda = 0$, which is equivalent to condition (21).

2.4. The integrated equations

As we mentioned in the Introduction, and for reasons that will become clear from the forthcoming analysis (see also the Discussion section), it is more convenient to recast the perturbation equations (13) (and the original nonlinear model (1) itself), in terms of the elastic deformation gradient variable $v := u_x$. Thus, formally, system of equations (13) can be written as

$$\begin{aligned} &\mu\theta v_{xx} + \mu v_{xt} + v_x - s\int_{-\infty}^{x} v(y,t)\,dy - \big(\tau'(\bar{c})c\big)_x = 0, \\ &c_t + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c = 0. \end{aligned} \tag{22}$$

We are interested in L^2 solutions to last system of equations. From its definition (assuming for instance $u \in H^1$) it is clear that v has mean zero (or "zero-mass") for each $t > 0$. Integrating in $x \in \mathbb{R}$ the elastic equation in (13) leads to the zero-mass property for u, i.e.,

$$\int_{\mathbb{R}} u(x,t)\,dx = 0,$$

for all $t > 0$. This is a necessary condition on u, for the antiderivative of u to belong to L^2, and it is naturally induced by the equations. Therefore, we expect the perturbation variable v to fulfill a double mean zero condition. The circumstances under which we are able to switch from one formulation to the other will depend on the naturally induced zero-mass conditions, as we shall see in Section 5 below.

In the same fashion, we express the original system (1) in terms of the deformation gradient as

$$\begin{aligned} &\mu v_{xt} + v_x + \tau(c)_x - s\int_{-\infty}^{x} v(y,t)\,dy = 0, \\ &c_t - Dc_{xx} - R(c) - \epsilon v = 0. \end{aligned} \tag{23}$$

We are now ready to state our main results.

[1] Here the order of a root λ_* of $D(\lambda)$ is $k \in \mathbb{N}$ if $(d/d\lambda)^j D(\lambda_*) = 0$ for all $0 \leqslant j \leqslant k-1$ and $(d/d\lambda)^k D(\lambda_*) \neq 0$.

3. Statement of results

In this paper we prove the following.

Theorem 1 *(Strong spectral stability). For $\epsilon > 0$ sufficiently small, traveling wave solutions $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ to (1) are strong spectrally stable. Moreover, $\lambda = 0$ is a simple isolated eigenvalue.*

Theorem 2 *(Linear stability and exponential decay). Suppose $\epsilon \geqslant 0$ is sufficiently small. Then there exists $\omega_0 > 0$ such that for any $(u_0, c_0) \in H^2 \times H^1$ there is a unique global mild solution $(v, c) \in C([0, +\infty); H^1 \times H^1)$ to Eqs. (22) (in the sense of Definition 5.13 below) with initial condition*

$$(v, c)(0) = (v_0, c_0) := (u_{0x}, c_0) \in H^1 \times H^1, \tag{24}$$

and an $\alpha_* \in \mathbb{R}$ such that

$$\left\| (v, c)(\cdot, t) - \alpha_* \left(\bar{u}^\epsilon_{xx}, \bar{c}^\epsilon_x\right)(\cdot) \right\|_{L^2 \times L^2} \lesssim e^{-\omega_0 t}, \tag{25}$$

for all $t > 0$.

Moreover, if $(u_0, c_0) \in (W^{1,1} \cap H^3) \times H^2$ then the solution above is a strong solution with $(v, c)(t) \in H^2 \times H^2$ for each $t > 0$, for which we can define the elastic variable

$$u(x, t) = \int_{-\infty}^{x} v(y, t)\, dy, \tag{26}$$

satisfying $u(t) \in H^3$ and

$$\int_{\mathbb{R}} u(x, t)\, dx = 0,$$

for each $t > 0$, such that $(u, c) \in C([0, +\infty); H^2 \times H^1)$, with $(u, c)(t) \in H^3 \times H^2$, is a strong solution to the linear system (13) with initial condition $(u, c)(0) = (u_0, c_0)$. In addition, the solution satisfies the decaying estimate

$$\left\| (u, c)(\cdot, t) - \alpha_* \left(\bar{u}^\epsilon_x, \bar{c}^\epsilon_x\right)(\cdot) \right\|_{L^\infty \times L^\infty} \to 0 \tag{27}$$

as $t \to +\infty$.

Theorem 3 *(Nonlinear orbital stability). Let $\epsilon \geqslant 0$ be sufficiently small. Then there exists $\eta_0 > 0$ such that for any $(\tilde{u}_0, \tilde{c}_0) \in H^2 \times H^1$ and any $\alpha_0 \in \mathbb{R}$ satisfying*

$$\left\| (\tilde{u}_{0x}, \tilde{c}_0)(\cdot) - \left(\bar{u}^\epsilon_x, \bar{c}^\epsilon\right)(\cdot + \alpha_0) \right\|_{H^1 \times H^1} < \eta \leqslant \eta_0, \tag{28}$$

then there exists a unique global solution $(\tilde{v}, \tilde{c}) \in C([0, +\infty); H^1 \times H^1)$ to Eqs. (23) and some $\alpha_\infty \in \mathbb{R}$, with $|\alpha_0 - \alpha_\infty| < C_1 \eta_0$, satisfying

$$\left\| (\tilde{v}, \tilde{c})(\cdot, t) - \left(\bar{u}^\epsilon_x, \bar{c}^\epsilon\right)(\cdot + \theta t + \alpha_\infty) \right\|_{H^1 \times H^1} \leqslant C \eta_0 e^{-\frac{1}{2}\omega_0 t} \to 0, \tag{29}$$

as $t \to +\infty$.

Moreover, if in addition $(\tilde{u}_0, \tilde{c}_0) \in (W^{1,1} \cap H^3) \times H^2$*, then there exists a unique global solution* $(\tilde{u}, \tilde{c}) \in C([0 + \infty); H^3 \times H^2)$ *to system* (1)*, such that*

$$\left\| (u, c)(\cdot, t) - \left(\bar{u}^\epsilon, \bar{c}^\epsilon\right)(\cdot + \theta t + \alpha_\infty) \right\|_{L^\infty \times L^\infty} \leqslant C\eta_0 e^{-\omega_0 t} \to 0, \tag{30}$$

as $t \to +\infty$*, for the same* $\alpha_\infty \in \mathbb{R}$*. Thus, the traveling waves are nonlinear orbitally* $(W^{1,1} \cap H^3) \times H^2 \to L^\infty \times L^\infty$ *stable. Here* $\omega_0 > 0$ *is the same decay rate as in Theorem* 2 *and depends only on* $\epsilon \geqslant 0$.

4. Spectral stability

This section is devoted to prove Theorem 1. First, we make an observation about the adjoint spectral problem which will be used throughout. Thereafter, our strategy will be to show the persistence of the spectral properties of the uncoupled waves with $\epsilon = 0$ to small values of $\epsilon > 0$.

4.1. Preliminaries: The adjoint system

Consider the associated adjoint equation

$$Y_x = -\mathbb{A}^\epsilon(x, \lambda)^* Y, \tag{31}$$

and the operators

$$\begin{aligned} \mathcal{T}^\epsilon(\lambda)^* : H^1\left(\mathbb{R}; \mathbb{C}^5\right) &\to L^2\left(\mathbb{R}; \mathbb{C}^5\right), \\ Y &\mapsto -Y_x - \mathbb{A}^\epsilon(x, \lambda)^* Y. \end{aligned} \tag{32}$$

Note that $\mathcal{T}^\epsilon(\lambda)^*$ is the Hilbert space adjoint operator of $\mathcal{T}^\epsilon(\lambda)$. One can explicitly write the coefficient matrices for the adjoint system (31) as

$$\mathbb{A}^\epsilon(x, \lambda)^* := -\begin{pmatrix} 0 & 0 & s/\mu\theta & 0 & 0 \\ 1 & 0 & 0 & 0 & -\epsilon/D \\ 0 & 1 & -(1 + \mu\lambda^*)/\mu\theta & 0 & 0 \\ 0 & 0 & \tau''(\bar{c})\bar{c}_x/\mu\theta & 0 & (\lambda^* - R'(\bar{c}))/D \\ 0 & 0 & \tau'(\bar{c})/\mu\theta & 1 & \theta/D \end{pmatrix}. \tag{33}$$

If for some $\lambda \in \mathbb{C}$, $\mathcal{T}^\epsilon(\lambda)$ is Fredholm with index i, then its adjoint $\mathcal{T}^\epsilon(\lambda)^*$ is Fredholm with index $-i$ [36]. Moreover, the adjoint system provides a characterization for the range of $\mathcal{T}^\epsilon(\lambda)$, as

$$V \in \mathcal{R}\left(\mathcal{T}^\epsilon(\lambda)\right) \subseteq L^2,$$

if and only if

$$\langle Y, V \rangle_{L^2} = \int_{\mathbb{R}} Y^* V \, dx = 0,$$

for all bounded $Y \in L^2$ solutions of the adjoint equation (31). (This observation was made by Palmer [30]; see also Remark 3.4 in [36].)

Remark 4.1. In view of the explicit form of the coefficients (33) for the adjoint system, it is worth noting that making

$$Y := (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{c}, \tilde{e})^\top, \tag{34}$$

Eq. (31) can be written component-wise as

$$\begin{aligned}
\tilde{u}_x &= -s\tilde{w}/\mu\theta,\\
\tilde{v}_x &= -\tilde{u} + \epsilon\tilde{e}/D,\\
\tilde{w}_x &= -\tilde{v} + \big(1+\mu\lambda^*\big)\tilde{w}/\mu\theta,\\
\tilde{c}_x &= -\tau''(\bar{c})\bar{c}_x\tilde{w}/\mu\theta - \big(\lambda^* - R'(\bar{c})\big)\tilde{e}/D,\\
\tilde{e}_x &= -\tau'(\bar{c})\tilde{w}/\mu\theta - \tilde{c} - \theta\tilde{e}/D.
\end{aligned} \tag{35}$$

Assume that the solutions to last system are regular enough. Therefore, substitution of Eqs. (35) and differentiation leads to a system of two equations for $\tilde{u}$ and $\tilde{e}$, namely

$$\begin{aligned}
\mu\theta\tilde{u}_{xxx} - \big(1+\mu\lambda^*\big)\tilde{u}_{xx} + s\tilde{u} - (s\epsilon/D)\tilde{e} &= 0,\\
D\tilde{e}_{xx} - \big(\lambda^* - R'(\bar{c})\big)\tilde{e} + \theta\tilde{e}_x - D\big(\tau'(\bar{c})/s\big)\tilde{u}_{xx} &= 0.
\end{aligned} \tag{36}$$

This implies that any sufficiently regular solution $(\tilde{u}, \tilde{e}) \in H^3 \times H^2$ to system (36) for some $\lambda \in \mathbb{C}$, determines a solution $Y \in H^1$ to (31), with $Y \in \ker \mathcal{T}^\epsilon(\lambda)^*$, and defined by means of (34), with $\tilde{v}$, $\tilde{w}$ and $\tilde{c}$ given by (35); explicitly,

$$\begin{aligned}
\tilde{v} &= (\mu\theta/s)\tilde{u}_{xx} + \big(\big(1+\mu\lambda^*\big)/\mu\theta\big)\tilde{u}_x,\\
\tilde{w} &= -(\mu\theta/s)\tilde{u}_x,\\
\tilde{c} &= \big(\tau'(\bar{c})/s\big)\tilde{u}_x - \tilde{e}_x - (\theta/D)\tilde{e}.
\end{aligned} \tag{37}$$

4.2. Spectral stability for $\epsilon = 0$

Consider the spectral problem (14) with $\epsilon = 0$,

$$\begin{aligned}
\mu\theta_0 u_{xxx} + (\mu\lambda + 1)u_{xx} - su - \big(\tau'(\bar{c})c\big)_x &= 0,\\
\lambda c + \theta_0 c_x - Dc_{xx} - R'(\bar{c})c &= 0,
\end{aligned} \tag{38}$$

where $(\bar{u}, \bar{c})$ denotes the wave solutions to (8) of Proposition 2.1. The spectral stability of such waves is a direct consequence of the results of Fife and McLeod [14], which show the asymptotic stability of the Nagumo (or bistable) calcium front. The spectral stability analysis of the Nagumo front can be found in [22] (see also [39] for a proof in weighted L^∞ norms, and [2] in the uniform norm). Let us state the main result of this section.

Lemma 4.2. *Traveling wave solutions $(\bar{u}, \bar{c})$ of system (8) are spectrally stable. Moreover, $\lambda = 0$ is an eigenvalue with algebraic multiplicity $a.m. = 1$.*

Proof. By the spectral analysis of Henry [22, pp. 128–131], it is well known that $\lambda = 0$ is the only eigenvalue with $\operatorname{Re}\lambda \geqslant 0$ of the linearized operator around the Nagumo front,

$$L_0 c := Dc_{xx} - \theta_0 c_x + R'(\bar{c})c, \tag{39}$$

densely defined in L^2 with domain $\mathcal{D}(L_0) = H^2(\mathbb{R})$. Moreover, $\lambda = 0$ is simple and with algebraic multiplicity $a.m. = 1$.

That $\lambda = 0$ is also an eigenvalue of (38) with eigenfunction $(\bar{u}_x, \bar{c}_x)$ is a direct consequence of the wave equations (8). Moreover, its geometric multiplicity (in the sense of Definition 2.7) is $g.m. = 1$. This follows from an elementary L^2 estimate: differentiate the equation for $\bar{u}$ in (8), multiply by any scalar α, and take the difference with the equation for u in (38); the result is

$$(u - \alpha\bar{u}_x)_x + \mu\theta_0(u - \alpha\bar{u}_x)_{xxx} - s(u - \alpha\bar{u}_x) = 0.$$

The L^2-product with $u - \alpha\bar{u}_x$ and integration by parts yield $\|u - \alpha\bar{u}_x\|_{L^2}^2 = 0$, which implies $u = \alpha\bar{u}_x$ a.e. Thus, $\bar{W} = (\bar{u}_x, \bar{u}_{xx}, \bar{u}_{xxx}, \bar{c}_x, \bar{c}_{xx})^\top$ is the only element in $\ker \mathcal{T}^0(0)$.

In order to verify that there are no other eigenvalues with $\operatorname{Re}\lambda \geqslant 0$, note that if (u, c) is an eigenfunction of (38) with non-trivial $c \in L^2$ then, because of stability of the Nagumo front, $L_0 c = \lambda c$ implies that $\operatorname{Re}\lambda < 0$. To preclude possible unstable eigenfunctions of the form $(u, 0)$ with non-trivial $u \in L^2$, notice that for $\operatorname{Re}\lambda \geqslant 0$, the only bounded L^2 solution to

$$(1 + \mu\lambda)u_{xx} + \mu\theta_0 u_{xxx} - su = 0 \tag{40}$$

is the trivial one $u = 0$ a.e. Indeed, take the L^2-product of (40) with u, integrate by parts, and take the real part to obtain

$$(1 + \mu \operatorname{Re}\lambda)\|u_x\|_{L^2}^2 + s\|u\|_{L^2}^2 = 0,$$

yielding $u \equiv 0$ a.e. if $\operatorname{Re}\lambda \geqslant 0$.

Hence, we conclude that $\lambda = 0$ is an isolated eigenvalue of (38) with $g.m. = 1$ and that there are no other eigenvalues with $\operatorname{Re}\lambda \geqslant 0$, proving strong spectral stability. As a consequence, the Evans function D^0 associated to (38) does not vanish for $\operatorname{Re}\lambda \geqslant 0$, $\lambda \neq 0$, and has an isolated zero at $\lambda = 0$.

Finally, to prove that the eigenvalue is simple, recall that the order of the isolated zero $\lambda = 0$ of D^0 coincides with its algebraic multiplicity ($a.m.$) as an eigenvalue of (38). Thus, in order to show that $a.m. = 1$ we need to compute $((d/d\lambda)D^0)_{|\lambda=0}$, which is given by

$$\frac{d}{d\lambda} D^0(0) = \Gamma_0 \hat{M}, \tag{41}$$

where $\Gamma_0 \neq 0$ is a non-vanishing factor measuring the orientation of the basis chosen to construct D^0, and $\hat{M}$ is the Melnikov integral computed with respect to the wave speed θ_0 (see [36, Section 4.2.1]). To compute $\hat{M}$, note that $\mathbb{A}^0(x, \lambda)$ is linear in λ and we can write $\mathbb{A}^0(x, \lambda) = \mathbb{A}^0(x, 0) + \lambda\tilde{\mathbb{A}}_1(x)$. Since the geometric multiplicity of $\lambda = 0$ as an eigenvalue of (38) is one, then there exists a unique bounded solution (up to constant multiples) $W \in L^2$ to $W_x = \mathbb{A}^0(x, 0)W$ and, because of the wave equations (8), W is in fact the derivative of the traveling wave $W = \bar{W}_x := (\bar{u}_x, \bar{u}_{xx}, \bar{u}_{xxx}, \bar{c}_x, \bar{c}_{xx})^\top$. Similarly, there exists a unique bounded solution to the adjoint equation (31) with $\lambda = 0$,

$$Y_x = -\mathbb{A}^0(x, 0)^* Y, \tag{42}$$

and the Melnikov integral $\hat{M}$ in (41) is given by $\hat{M} = \langle Y, \tilde{\mathbb{A}}_1 W\rangle_{L^2}$ (see [36]).

To solve (42) denote $Y = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{c}, \tilde{e})^\top$, and use (33) with $\epsilon = 0$ and $\lambda = 0$ to find that system (42) amounts to solving

$$\begin{aligned}
\tilde{u}_x &= -(s/\mu\theta_0)\tilde{w}, \\
\tilde{v}_x &= -\tilde{u}, \\
\tilde{w}_x &= -\tilde{v} + \tilde{w}/\mu\theta_0, \\
\tilde{c}_x &= -\big(\tau''(\bar{c})\bar{c}'/\mu\theta_0\big)\tilde{w} + R'(\bar{c})\tilde{e}/D, \\
\tilde{e}_x &= -\big(\tau'(\bar{c})/\mu\theta_0\big)\tilde{w} - \tilde{c} - \theta_0\tilde{e}/D.
\end{aligned}$$

From the results in [16, pp. 57–58] it is known that the only solutions to the equations for the elastic variables $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ in the last system are $\tilde{u} = \tilde{v} = \tilde{w} \equiv 0$, and the remaining calcium concentration equations $\tilde{c}_x = R'(\bar{c})\tilde{e}/D$, and $\tilde{e}_x = -\tilde{c} - \theta_0\tilde{e}/D$, reduce to solving the scalar equation

$$D\tilde{e}_{xx} + \theta_0\tilde{e}_x + R'(\bar{c})\tilde{e} = 0.$$

By Lemma 4 in [16], we know that the only bounded solution to the last equation is $\tilde{e} := \bar{c}_x(-x)$. Therefore we can easily verify that $Y^*\tilde{\mathbb{A}}_1 W = (1/D)\bar{c}_x(-x)\bar{c}_x(x)$, and the Melnikov integral is given by

$$\hat{M} = \frac{1}{D}\int_{-\infty}^{+\infty} \bar{c}'(-x)\bar{c}'(x)\,dx > 0, \tag{43}$$

by monotonicity of the calcium profile. Therefore, $(d/d\lambda)D^0(0) \neq 0$ and the algebraic multiplicity is $a.m. = 1$. The lemma is now proved. $\square$

We summarize the results of this section in the following

Corollary 4.3. *The Evans function associated to the spectral problem* (38) *satisfies*

$$D^0(\lambda) \neq 0, \quad \textit{for } \operatorname{Re}\lambda \geqslant 0, \quad \lambda \neq 0. \tag{44}$$

Moreover, D^0 has an isolated zero of order one at $\lambda = 0$.

4.3. Boundedness of the point spectrum

The following lemma shows that the point spectrum of problem (14) is uniformly bounded in a connected region properly containing the unstable half plane $\{\operatorname{Re}\lambda \geqslant 0\}$.

Lemma 4.4. *For each $\epsilon \geqslant 0$, the point spectrum of problem* (14) *in the region $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant -\frac{1}{2\mu}\}$ is uniformly bounded.*

Proof. Assume $\lambda \in \sigma_{\mathrm{pt}} \cap \{\operatorname{Re}\lambda \geqslant -\frac{1}{2\mu}\}$, with $W \in \ker\mathcal{T}^\epsilon(\lambda)$ and fixed $\epsilon \geqslant 0$. Thus, $W = (u, u_x, u_{xx}, c, c_x)^\top$ with $(u, c) \in H^3 \times H^2$. Take the L^2-product of the calcium equation of (14) with c and integrate by parts; the result is

$$\lambda\|c\|_{L^2}^2 + D\|c_x\|_{L^2}^2 + \theta\langle c, c_x\rangle_{L^2} = \epsilon\langle c, u_x\rangle_{L^2} + \int_{\mathbb{R}} R'(\bar{c})|c|^2\,dx. \tag{45}$$

Take the real part of (45) to obtain

$$(\operatorname{Re}\lambda)\|c\|_{L^2}^2 \leqslant \frac{1}{2}\epsilon\|u_x\|_{L^2}^2 + \left(M_1 + \frac{1}{2}\epsilon\right)\|c\|_{L^2}^2, \tag{46}$$

where $M_1 := \max_{c\in[0,1]}|R'(c)| > 0$. Take now the L^2-product of the elastic equation of (14), integrate by parts and take its real part to arrive at

$$s\|u\|_{L^2}^2 + (1 + \mu\operatorname{Re}\lambda)\|u_x\|_{L^2}^2 = \operatorname{Re}\langle u_x, \tau'(\bar{c})c\rangle_{L^2} \leqslant M_2\|u_x\|_{L^2}\|c\|_{L^2}, \tag{47}$$

where $M_2 := \max_{c\in[0,1]} |\tau'(c)| > 0$. Since by assumption $0 < (1+\mu \operatorname{Re}\lambda)^{-1} \leqslant 2$, Eq. (47) and $s > 0$ yield the estimate

$$\|u_x\|_{L^2} \leqslant (1+\mu \operatorname{Re}\lambda)^{-1} M_2 \|u_x\|_{L^2} \leqslant 2M_2\|c\|_{L^2}. \tag{48}$$

Substitution of (48) into (46) yields

$$\left(\operatorname{Re}\lambda - \left(M_1 + \left(2M_2^2 + \frac{1}{2}\right)\epsilon\right)\right)\|c\|_{L^2} \leqslant 0.$$

This shows that for each $\epsilon \geqslant 0$ we have the uniform bound

$$\operatorname{Re}\lambda \leqslant K_1(\epsilon) := M_1 + \left(2M_2^2 + \frac{1}{2}\right)\epsilon, \tag{49}$$

with uniform constants $M_1, M_2 > 0$. Notice that $K_1(\epsilon) > 0$ is also uniformly bounded in $\epsilon \in [0,1]$.

To obtain the bound for $|\operatorname{Im}\lambda|$ take the imaginary part of (45) to get

$$|\operatorname{Im}\lambda|\|c\|_{L^2} \leqslant \theta\|c_x\|_{L^2} + \epsilon\|u_x\|_{L^2}. \tag{50}$$

Observe that since $\operatorname{Re}\lambda \geqslant -\frac{1}{2\mu}$, taking the real part of Eq. (45) yields

$$D\|c_x\|_{L^2}^2 \leqslant \frac{1}{2}\epsilon\|u_x\|_{L^2}^2 + \left(M_1 + \frac{1}{2}\epsilon + \frac{1}{2\mu}\right)\|c\|_{L^2}^2. \tag{51}$$

Combine (51) with (48), and substitute into (50) to obtain

$$\big(|\operatorname{Im}\lambda| - K_2(\epsilon)\big)\|c\|_{L^2} \leqslant 0,$$

with

$$K_2(\epsilon) := \theta D^{-1/2}\left(M_1 + \frac{1}{2\mu} + \left(\frac{1}{2} + 2M_2^2\right)\epsilon\right)^{1/2} + 2\epsilon M_2 > 0,$$

for each $\epsilon \geqslant 0$. Note that K_2 is an increasing function of $\epsilon \geqslant 0$, for which, $0 < K_2(0) \leqslant K_2(\epsilon) \leqslant K_2(1)$ for all $\epsilon \in [0,1]$. Therefore we get the uniform bound

$$|\operatorname{Im}\lambda| \leqslant K_2(\epsilon). \tag{52}$$

Estimates (49) and (52) provide the result. □

4.4. Hyperbolicity and consistent splitting

We now look at the asymptotic coefficients (17) of the coupled problem with $\epsilon \geqslant 0$. From the existence result (Proposition 2.3) we know that

$$\theta^{\epsilon} = \theta_0 + \delta(\epsilon), \tag{53}$$

with $\delta(\epsilon) = o(1)$ as $\epsilon \to 0^+$. The function $\delta(\cdot)$ will determine the rate of convergence of the family of Evans functions. Plugging (53) into the coefficients (17) and expanding θ^ϵ around θ_0 we obtain $\mathbb{A}^\epsilon_\pm(\lambda) = \mathbb{A}^0_\pm(\lambda) + \mathbb{S}^\epsilon_\pm(\lambda)$, with

$$\mathbb{S}^\epsilon_\pm := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ s\mathcal{O}(\delta(\epsilon))/\mu & 0 & -(1+\mu\lambda)\mathcal{O}(\delta(\epsilon))/\mu & 0 & \tau'(n)\mathcal{O}(\delta(\epsilon))/\mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon/D & 0 & 0 & \mathcal{O}(\delta(\epsilon))/D \end{pmatrix}. \tag{54}$$

$\mathbb{A}^\epsilon_\pm(\lambda)$ is linear (analytic) in λ and continuous in ϵ. The perturbation matrix $\mathbb{S}^\epsilon_\pm$ converges to zero at a rate $\mathcal{O}(|\delta(\epsilon)| + \epsilon)$, and has the form

$$\mathbb{S}^\epsilon_\pm(\lambda) = -\mathcal{O}\big(\delta(\epsilon)\big) \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \lambda & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + \mathcal{O}\big(\delta(\epsilon) + \epsilon\big),$$

so that $|\mathbb{S}^\epsilon_\pm(\lambda)| \leqslant \mathcal{O}(|\delta(\epsilon)| + \epsilon)(1 + |\lambda|)$. Let us denote the characteristic polynomial of $\mathbb{A}^\epsilon_\pm$ as

$$\pi^\epsilon_\pm(\kappa) := \det\big(\mathbb{A}^\epsilon_\pm(\lambda) - \kappa I\big). \tag{55}$$

Notice that the coefficients (17) can be expressed as

$$\mathbb{A}^\epsilon_\pm := \begin{pmatrix} \mathbb{A}_1 & \mathbb{A}_2 \\ \mathbb{A}_3 & \mathbb{A}_4 \end{pmatrix},$$

with

$$\begin{aligned} \mathbb{A}_1 &:= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta \end{pmatrix}, \\ \mathbb{A}_2 &:= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \tau'(n)/\mu\theta \end{pmatrix}, \\ \mathbb{A}_3 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon/D & 0 \end{pmatrix}, \\ \mathbb{A}_4 &:= \begin{pmatrix} 0 & 1 \\ (\lambda - R'(n))/D & \theta/D \end{pmatrix}. \end{aligned} \tag{56}$$

Remark 4.5. 1. For each $a \in \mathbb{R}$, the λ-roots of the equation

$$\pi^\epsilon_\pm(ia) = \det\big(\mathbb{A}^\epsilon_\pm(\lambda) - iaI\big) = 0$$

define algebraic curves $\{\lambda^\epsilon_{\pm,j}(a)\colon a \in \mathbb{R}\}$, $j = 1, 2$, in the complex plane. Denote by Ω^ϵ_j the open subset of $\mathbb{C}$ bounded on the left by the two curves $\lambda^\epsilon_{+,j}(a)$ and $\lambda^\epsilon_{-,j}(a)$ for each j. Then define the following open connected subset of $\mathbb{C}$

$$\tilde{\Omega}^\epsilon := \bigcap_j \Omega^\epsilon_j.$$

$\tilde{\Omega}^\epsilon$ is called the region of *consistent splitting* of system (15) for each $\epsilon \geqslant 0$ (see [1]). By connectedness, the dimensions of the stable/unstable spaces of $\mathbb{A}^\epsilon_\pm(\lambda)$ do not change for $\lambda \in \tilde{\Omega}^\epsilon$.

2. For instance, let us look at the algebraic curves $\lambda^0_{\pm,j}(a)$ when $\epsilon = 0$. In this case $\mathbb{A}_3 = 0$ and the characteristic polynomial is

$$\begin{aligned}\pi^0_\pm(ia) &= \det(\mathbb{A}_1 - iaI)\det(\mathbb{A}_4 - iaI)\\ &= \big(a^2\big(ia + (1+\mu\lambda)/\mu\theta_0\big) + s/\mu\theta_0\big)\big(ia(ia - \theta_0/D) - \big(\lambda - R'(n)\big)/D\big) = 0.\end{aligned}$$

Thus, the curves that bound $\tilde{\Omega}^0$ are

$$\lambda^0_{\pm,1}(a) = R'(n) - i\theta_0 a - Da^2, \tag{57}$$

$$\lambda^0_2(a) = -\big(s/\mu a^2 + 1/\mu + i\mu\theta_0 a\big), \quad a \neq 0. \tag{58}$$

Clearly $\operatorname{Re}\lambda^0_{\pm,1}(a) \leqslant R'(n) < 0$ and $\operatorname{Re}\lambda^0_2(a) \leqslant -1/\mu < 0$ for all values of a, and therefore we have the proper inclusion $\{\operatorname{Re}\lambda \geqslant 0\} \subset \tilde{\Omega}^0$.

In view of last remark, define

$$\Lambda_0 := \min\big\{\big|R'(1)\big|, \big|R'(0)\big|, 1/\mu\big\} > 0 \tag{59}$$

and consider the following open connected region in $\mathbb{C}$

$$\Omega := \left\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\frac{1}{2}\Lambda_0\right\}. \tag{60}$$

Remark 4.6. Thanks to Lemma 4.4 the point spectrum of problem (14) is uniformly bounded in Ω for each fixed $\epsilon \geqslant 0$. The set Ω properly contains the unstable half space $\{\operatorname{Re}\lambda \geqslant 0\}$.

Denote $S^\epsilon_\pm(\lambda)$ (resp. $U^\epsilon_\pm(\lambda)$) as the stable (resp. unstable) eigenspaces of $\mathbb{A}^\epsilon_\pm(\lambda)$. The following lemma is the main observation of this section.

Lemma 4.7. *For all $\lambda \in \Omega$ and all $\epsilon \geqslant 0$ sufficiently small, the coefficient matrices $\mathbb{A}^\epsilon_\pm(\lambda)$ have no center eigenspace and* $\dim U^\epsilon_\pm(\lambda) = 2$ *and* $\dim S^\epsilon_\pm(\lambda) = 3$.

Proof. First observe that $\mathbb{A}_4$ has no center eigenspace for all $\epsilon \geqslant 0$ and all $\lambda \in \Omega$, as

$$\begin{aligned}\operatorname{Re}\det(\mathbb{A}_4 - ia) &= \operatorname{Re}\big(ia(ia - \theta/D) - \big(\lambda - R'(n)\big)/D\big)\\ &= -\big(a^2 + \big(\operatorname{Re}\lambda - R'(n)\big)/D\big) < 0,\end{aligned}$$

for all $a \in \mathbb{R}$ and all $\epsilon \geqslant 0$. We denote then $p^\epsilon(ia) := \det(\mathbb{A}_4 - ia) \neq 0$, $a \in \mathbb{R}$.

First, suppose that $\tau'(n) = 0$. This implies $\mathbb{A}_2 = 0$. Suppose that (55) has a purely imaginary root ia with $a \in \mathbb{R}$. Hence

$$\pi^\epsilon_\pm(ia) = p^\epsilon(ia)\det(\mathbb{A}_1 - ia) = 0$$

if and only if

$$\det(\mathbb{A}_1 - ia) = ia^3 + a^2(1+\mu\lambda)/\mu\theta + s\mu\theta = 0. \tag{61}$$

Taking the real part of (61) leads to a contradiction with $\lambda \in \Omega$, namely

$$0 = a^2(1+\mu \operatorname{Re}\lambda) + s > 0,$$

for all a. Therefore, in the case $\tau'(n) = 0$, $\mathbb{A}_\pm^\epsilon$ has no center eigenspace for all $\epsilon \geqslant 0$.

By continuity and connectedness of the region of consistent splitting Ω, the dimensions of the stable (resp. unstable) eigenspaces $S_\pm^\epsilon$ (resp. $U_\pm^\epsilon$) must remain constant on Ω. To compute the dimensions it suffices to set $\lambda = \eta \in \mathbb{R}^+$ and let $\eta \to +\infty$. The characteristic polynomial becomes

$$\begin{aligned}\pi_\pm^\epsilon(\kappa) &= \det\bigl(\mathbb{A}_1(\eta) - \kappa\bigr)\det\bigl(\mathbb{A}_4(\eta) - \kappa\bigr)\\ &= \bigl(-\kappa^2\bigl(\kappa + (1+\mu\eta)/\mu\theta\bigr) + s/\mu\theta\bigr)\bigl(\kappa(\kappa - \theta/D) - \bigl(\eta - R'(n)\bigr)/D\bigr).\end{aligned} \tag{62}$$

The quadratic factor in (62) produces two roots

$$\kappa_{1,2}^\epsilon = \frac{1}{2D}\bigl(\theta \pm \sqrt{\theta^2 + 4D\bigl(\eta - R'(n)\bigr)}\bigr).$$

Since $\eta - R'(n) > 0$, $D > 0$, then clearly one root is positive, say $\kappa_1^\epsilon > 0$, and the other is negative, $\kappa_2^\epsilon < 0$. The cubic factor in (62), namely $H(\kappa) := s - \kappa^2(\kappa\mu\theta + (1+\mu\lambda))$, produces three roots; a local maximum of H occurs at $\kappa = 0$, with value $H(0) = s > 0$, and a local minimum occurs at $\kappa_m = -2(1+\mu\eta)/(3\mu\theta)$ with value $H(\kappa_m) = s - 4(1+\mu\eta)^3/(27\mu^2\theta^2)$. Letting η be sufficiently large, we can make the value of H at the local minimum negative, so that we have 2 real negative roots $\kappa_{3,4}^\epsilon < 0$ and one positive real root $\kappa_5^\epsilon > 0$. This shows that, on Ω, there exist 3 roots $\kappa_{2,3,4}^\epsilon(\lambda)$ with $\operatorname{Re}\kappa < 0$, and 2 roots $\kappa_{1,5}^\epsilon(\lambda)$ with $\operatorname{Re}\kappa > 0$. Thus $\dim S_\pm^\epsilon = 3$ and $\dim U_\pm^\epsilon = 2$, and the result holds when $\tau'(n) = 0$.

Consider now what happens when $\tau'(n) \neq 0$. For fixed $\lambda \in \Omega$, the dimensions will change only if the eigenvalues cross the imaginary axis, that is, if for some value of $\tau'(n)$ there exists $a \in \mathbb{R}$ such that $\pi_\pm^\epsilon(ia) = 0$. Note that $\mathbb{A}_4 - ia$ is invertible as $p^\epsilon(ia) \neq 0$ for all $a \in \mathbb{R}$. Thus one can write

$$\begin{aligned}\pi_\pm^\epsilon(ia) &= \det\begin{pmatrix}\mathbb{A}_1 - ia & \mathbb{A}_2\\ \mathbb{A}_3 & \mathbb{A}_4 - ia\end{pmatrix}\\ &= \det\left(\begin{pmatrix} I & \mathbb{A}_2\\ 0 & \mathbb{A}_4 - ia\end{pmatrix}\begin{pmatrix}\mathbb{A}_1 - ia - \mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 & 0\\ (\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 & I\end{pmatrix}\right)\\ &= \det(\mathbb{A}_4 - ia)\det\bigl(\mathbb{A}_1 - ia - \mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3\bigr).\end{aligned}$$

Perform the matrix computations to find that

$$(\mathbb{A}_4 - ia)^{-1} = p^\epsilon(ia)^{-1}\begin{pmatrix}\theta/D - ia & -1\\ (R'(n) - \lambda)/D & -ia\end{pmatrix},$$

$$\mathbb{A}_2(\mathbb{A}_4 - ia)^{-1}\mathbb{A}_3 = p^\epsilon(ia)^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & ia\epsilon\tau'(n)/(\mu\theta D) & 0\end{pmatrix},$$

and

$$\pi_\pm^\epsilon(ia) = p^\epsilon(ia)\det\begin{pmatrix}-ia & 1 & 0\\ 0 & -ia & 1\\ s/\mu\theta & -ia\epsilon\tau'(n)/(\mu\theta D p^\epsilon(ia)) & -ia - (1+\mu\lambda)/\mu\theta\end{pmatrix}. \tag{63}$$

Since $p^\epsilon(ia) \neq 0$, then $\pi^\epsilon_\pm(ia) = 0$ if and only if

$$i\mu\theta a^3 + s + a^2\left(\frac{\epsilon\tau'(n)}{Dp^\epsilon(ia)} + 1 + \mu\lambda\right) = 0.$$

In view of $s > 0$ and $\operatorname{Re}\lambda > -1/\mu$, taking real part of last equation and for $\epsilon \geqslant 0$ sufficiently small we arrive at

$$0 = s + \frac{\epsilon\tau'(n)\operatorname{Re} p^\epsilon(ia)}{D|p^\epsilon(ia)|^2} + 1 + \mu\operatorname{Re}\lambda > 0,$$

which is a contradiction. This shows that the signs of the real parts of the eigenvalues of $\mathbb{A}^\epsilon_\pm$ are independent of $\tau'(n)$ for each $\lambda \in \Omega$ and $\epsilon \geqslant 0$ sufficiently small, yielding the result. □

An immediate consequence of hyperbolicity and consistent splitting (the dimensions of the stable/unstable spaces at both ends $x = \pm\infty$ agree) is the stability of the essential spectrum.

Corollary 4.8. *For $\epsilon \geqslant 0$ sufficiently small, the essential spectrum of* (14) *is contained in the stable half plane.*

Proof. Fix $\lambda \in \Omega$ and $\epsilon \geqslant 0$ in a neighborhood of zero. $\mathbb{A}^\epsilon_+(\lambda)$ and $\mathbb{A}^\epsilon_-(\lambda)$ being hyperbolic, by standard exponential dichotomies theory [6] (see also Theorem 3.3 in [36]), system (15) has exponential dichotomies on both $x \in \mathbb{R}^+$ and $x \in \mathbb{R}^-$, with Morse indices $i_+(\lambda) = \dim U^\epsilon_+(\lambda) = 2$ and $i_-(\lambda) = \dim U^\epsilon_-(\lambda) = 2$, respectively. Henceforth, Palmer's results (Lemma 3.4 in [30]; see also Theorem 3.2 in [36]) imply that the operators $\mathcal{T}^\epsilon(\lambda)$ are Fredholm with index

$$\operatorname{ind}\mathcal{T}^\epsilon(\lambda) = i_+(\lambda) - i_-(\lambda) = 0,$$

showing that $\Omega \subset \mathbb{C}\backslash\sigma_{\mathrm{ess}}$, or in other words, that $\sigma_{\mathrm{ess}} \subset \mathbb{C}\backslash\Omega \subset \{\operatorname{Re}\lambda < 0\}$. □

Remark 4.9. Note that Ω is contained in the complement of the essential spectrum, and that $\{\operatorname{Re}\lambda \geqslant 0\} \subset \Omega$. Likewise, the set Ω is, in turn, properly contained in the region of consistent splitting $\tilde{\Omega}^\epsilon$ for each $\epsilon > 0$ sufficiently small. This set was chosen such that the coefficient asymptotic matrices are hyperbolic, and that the dimensions of the respective unstable spaces agree for all ϵ in a neighborhood of zero. Also notice that there is a *uniform* positive spectral gap between the essential spectrum and the imaginary axis for all $0 \leqslant \epsilon \ll 1$ sufficiently small.

Remark 4.10. A closer look at the algebraic curves that bound the essential spectrum yields further spectral information. For instance, continued inspection of the characteristic polynomial (63) leads to

$$\begin{aligned}
\mu\theta D\pi^\epsilon_\pm(ia) = {} & -i\mu\theta Da^5 + \big(\mu\theta^2 - (1+\mu\lambda)D\big)a^4 \\
& - i\theta\big(\big(\lambda - R'(n)\big)\mu + 1 + \mu\lambda\big)a^3 \\
& + \big(\epsilon\tau'(n) - (1+\mu\lambda)\big(\lambda - R'(n)\big) - sD\big)a^2 \\
& - is\theta a - \big(\lambda - R'(n)\big)s.
\end{aligned} \tag{64}$$

As we have pointed out, the λ-roots of (64) define algebraic curves for each $\epsilon \geqslant 0$. One can, however, regard these roots $\lambda^\epsilon_{\pm,j}(a)$ as algebraic functions of ϵ for each fixed $a \in \mathbb{R}$. If $a = 0$ then $\lambda^\epsilon_{\pm,j}(a) = R'(n) < 0$, which is a point contained in Ω, uniformly for each $\epsilon \geqslant 0$. Fixing $a \in \mathbb{R}$, $a \neq 0$, by the theory of algebraic functions [25], the curves $\lambda^\epsilon_{\pm,j}(a)$ are either single-valued analytic functions of ϵ near zero, or admit algebraic branches with converging Puiseux series expansions with fractional powers of ϵ; in both cases $\epsilon = 0$ is a critical point. Notice that the highest order λ-monomial in

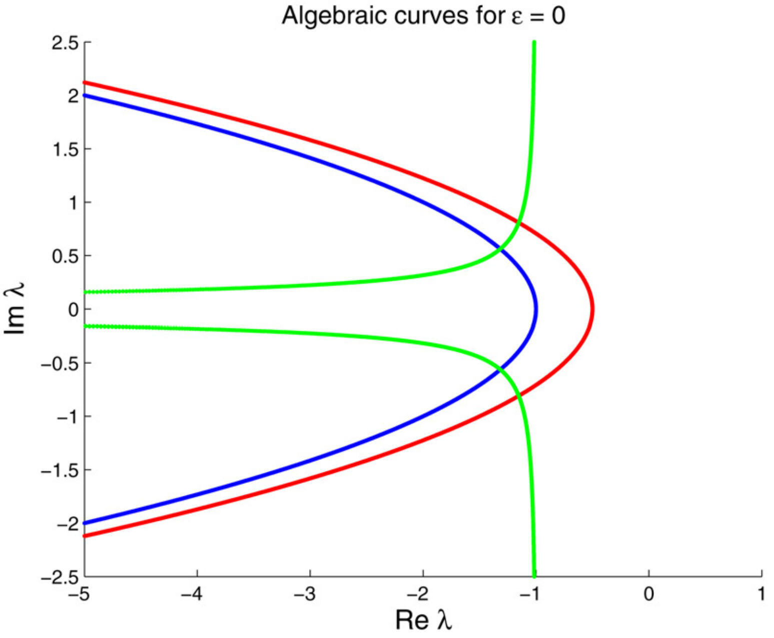

Fig. 2. Plots of the algebraic curves (57) and (58), delimiting on the right the location of the essential spectrum for the system with $\epsilon = 0$, for parameter values of $\mu = 1$, $\theta = 1$, $D = 1$ and $R'(1) = -1$, $R'(0) = -\frac{1}{2}$, with $s = 0.1 < 2\theta/\mu\sqrt{27}$. The algebraic curves $\lambda^{\epsilon}_{1,\pm}(a)$ and $\lambda^{\epsilon}_{2}(a)$, $a \in \mathbb{R}$, for $\epsilon \sim 0^+$ are regular perturbations of the curves here depicted. Note that, for $\epsilon \geqslant 0$ sufficiently small, there is a uniform spectral gap between the essential spectrum and $\lambda = 0$, of order $\mathcal{O}(\Lambda_0)$. Moreover, the "sectoriality" property does not hold (see Remark 4.10).

(64) is $-\mu a^2\lambda^2$, with non-zero coefficient $-\mu a^2 \neq 0$. Therefore, in either case, the series expansions *do not* contain negative powers of ϵ and the curves $\lambda^{\epsilon}_{\pm,j}(a)$ are regular perturbations, for all $\epsilon > 0$ sufficiently small, of the curves $\lambda^{0}_{\pm,j}(a)$ for $\epsilon = 0$ (see (57) and (58) in Remark 4.5.2).

Therefore, the curves $\lambda^{0}_{\pm,j}(a)$ are a good approximation of the limiting curves bounding the essential spectrum for $\epsilon > 0$ small. Fig. 2 shows the computed algebraic curves (57) and (58) for $\epsilon = 0$ with specific parameter values. The spectral gap between the essential spectrum and the origin can be observed. Moreover, the asymptotic behavior of the curve (58) precludes the operators $\mathcal{T}^{\epsilon}(\lambda)$ to fulfill the "sectoriality" property that for some $\beta \in (0, \pi/2)$, $M \geqslant 1$ and $\alpha \in \mathbb{R}$ the sector

$$S_{\alpha,\beta} = \left\{\lambda \in \mathbb{C}\colon \left|\arg(\lambda - \alpha)\right| < \pi/2 + \beta,\ \lambda \neq \alpha\right\}$$

belongs to the resolvent set ρ of Definition 2.6, and

$$\left\|\mathcal{T}^{\epsilon}(\lambda)^{-1}\right\| \leqslant \frac{M}{|\lambda - \alpha|}, \quad \text{for all } \lambda \in S_{\alpha,\beta}.$$

This property is not satisfied for ϵ sufficiently small. This observation suggests that the most we expect is the solutions of the linear system (13) to generate a C_0-semigroup (as we shall prove in Section 5), and not an analytic semigroup.

4.5. *Convergence of Evans functions*

Consider the family of first order systems (15), with $\lambda \in \Omega$ and ϵ varying within a set $\mathcal{V} = [0, \epsilon_0]$, where $\epsilon_0 > 0$ is chosen sufficiently small such that the conclusions of Proposition 2.3, and of Lemmas 2.4 and 4.7, hold. If we regard coefficients (16) as functions from (λ,ϵ) into $L^\infty(\mathbb{R})$, then they are analytic in λ (linear), and continuous in ϵ (this follows by continuity of θ and $(\bar u^\epsilon, \bar c^\epsilon)$ on ϵ, by construction [16]). Moreover, $\mathbb{A}^\epsilon(\cdot,\lambda)$ approach exponentially to limits $\mathbb{A}_\pm$ as $x \to \pm\infty$, with uniform exponential decay estimates

$$\left|\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm\right| \leqslant Ce^{-|x|/C_1}, \quad \text{for } x \gtrless 0 \tag{65}$$

on compact subsets of $\Omega \times \mathcal{V}$. In addition, on $\Omega \times \mathcal{V}$ the limiting coefficient matrices $\mathbb{A}^\epsilon_+$ and $\mathbb{A}^\epsilon_-$ are both hyperbolic and the dimensions of their unstable subspaces agree. Finally, the geometric separation hypothesis of Gardner and Zumbrun [17] holds trivially (continuous limits of $S^\epsilon_\pm$ and $U^\epsilon_\pm$ along λ-rays, $\lambda = \rho\lambda_0$ as $\rho \to 0^+$, $\lambda_0 \in \Omega$; see condition (A2) in [34]) because of hyperbolicity of the coefficients (with same dimensions) at $\lambda = 0$ as discussed in [16].

Hence, systems (15) belong to the generic class considered in Section 2 of [34], for which the convergence of approximate flows (Proposition 2.4 in [34]) applies. The proof of Theorem 1 is a consequence of the following result.

Lemma 4.11. *Suppose $(\lambda,\epsilon) \in \Omega \times \mathcal{V}$. Then the asymptotic spaces U^ϵ_- and S^ϵ_+ converge uniformly in angle to U^0_-, S^0_+ as $\epsilon \to 0^+$, with rate $\eta(\epsilon) = \mathcal{O}(\epsilon + |\delta(\epsilon)|)$; that is, for all $\epsilon \in \mathcal{V}$ their spanning bases satisfy*

$$\left|v^\epsilon_{j\pm} - v^0_{j\pm}\right| \leqslant \eta(\epsilon). \tag{66}$$

Moreover, the coefficient matrices (16) *converge uniformly exponentially to limiting values $\mathbb{A}^0$, in the sense that, for all $\epsilon \in \mathcal{V}$,*

$$\left|\left(\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm\right) - \left(\mathbb{A}^0 - \mathbb{A}^0_\pm\right)\right| \leqslant C_2\eta(\epsilon)e^{-|x|/C_1}. \tag{67}$$

Proof. To prove (66), it suffices to show that the projections

$$\mathcal{P}^\epsilon_\pm(\lambda) := \frac{1}{2\pi i}\oint_{\Gamma^\epsilon_\pm}\left(z - \mathbb{A}^\epsilon_\pm\right)^{-1} dz$$

are uniformly bounded as $\epsilon \to 0^+$ in a closed Ω-neighborhood of λ, with

$$\mathcal{P}^\epsilon_\pm = \mathcal{P}^0_\pm + \mathcal{O}\left(\epsilon + \left|\delta(\epsilon)\right|\right).$$

Here Γ^ϵ_- (resp. Γ^ϵ_+) is a rectifiable contour enclosing the unstable (resp. stable) eigenvalues of $\mathbb{A}^\epsilon_-$ (resp. $\mathbb{A}^\epsilon_+$). Compactness then gives a uniform resolvent bound

$$\left|R^0_\pm(z)\right| := \left|\left(\mathbb{A}^0_\pm - zI\right)^{-1}\right| \leqslant C \quad \text{on } \Gamma^\epsilon_\pm.$$

Here C depends only on $|\mathbb{A}^0_\pm|$. Recall that $\mathbb{A}^\epsilon_\pm = \mathbb{A}^0_\pm + \mathbb{S}^\epsilon_\pm$, with perturbation matrix $\mathbb{S}^\epsilon_\pm$ given by (54). Whence one may expand

$$\begin{aligned}
\left(\mathbb{A}^\epsilon_\pm - zI\right)^{-1} &= \left(\mathbb{A}^0_\pm - zI + \mathbb{S}^\epsilon_\pm\right)^{-1} \\
&= \left(\left(\mathbb{A}^0_\pm - zI\right)\left(I + \left(\mathbb{A}^0_\pm - zI\right)^{-1}\mathbb{S}^\epsilon_\pm\right)\right)^{-1} \\
&= \left(I + R^0_\pm(z)\mathbb{S}^\epsilon_\pm\right)^{-1}R^0_\pm(z) \\
&= \left(I + \mathcal{O}\left(\left|\delta(\epsilon)\right| + \epsilon\right)\right)R^0_\pm(z),
\end{aligned}$$

where $\eta(\epsilon) := \mathcal{O}(|\delta(\epsilon)| + \epsilon)$ depends on $|\mathbb{A}^0|$. We then get

$$\mathcal{P}_\pm^\epsilon = \mathcal{P}_\pm^0 + \mathcal{O}\bigg(\int_{\Gamma_\pm^0} \big|R_\pm^0(z)\big|\big|\eta(\epsilon)\big|\,dz\bigg) = \mathcal{P}_\pm^0 + \mathcal{O}\big(\eta(\epsilon)\big),$$

as claimed.

The second assertion (67) is an immediate consequence of exponential decay (Lemma 2.4). Indeed, a direct computation shows that

$$\begin{aligned}&\big|\big(\mathbb{A}^\epsilon - \mathbb{A}_\pm^\epsilon\big) - \big(\mathbb{A}^0 - \mathbb{A}_\pm^0\big)\big| \\ &\quad = \mathcal{O}\big(\big(\big|\bar{c}_x^\epsilon\big| + \big|\tau'\big(\bar{c}^\epsilon\big) - \tau'(n)\big|\big)/\theta\big) + \mathcal{O}\big(\big(|\bar{c}_x| + \big|\tau'(\bar{c}) - \tau'(n)\big|\big)/\theta_0\big),\end{aligned}$$

and using (11),

$$\begin{aligned}\big|\tau'\big(\bar{c}^\epsilon\big) - \tau'(1)\big| &= \mathcal{O}\big(\big|\bar{c}^\epsilon - 1\big|\big) = \mathcal{O}\big(e^{-|x|/C_1}\big),\\ \big|\tau'\big(\bar{c}^\epsilon\big) - \tau'(0)\big| &= \mathcal{O}\big(\big|\bar{c}^\epsilon\big|\big) = \mathcal{O}\big(e^{-|x|/C_1}\big),\\ \big|\bar{c}_x^\epsilon\big| &= \mathcal{O}\big(e^{-|x|/C_1}\big),\end{aligned}$$

uniformly in $\epsilon \in \mathcal{V}$ (including $\epsilon = 0$), and $1/\theta = 1/\theta_0 + \mathcal{O}(\delta(\epsilon))$; therefore (67) follows directly. □

Remark 4.12. The convergence result in [34] is more general than what we actually need for the applications of this paper. Indeed, in contrast to the viscous shocks case, here S_+^0 and U_-^0 do correspond to actual stable and unstable subspaces for the limiting matrices $\mathbb{A}_\pm^0$, which are hyperbolic for $\lambda \in \Omega$. Moreover, hypotheses (67) and (66) hold for regular perturbations around $\epsilon = 0$ of the profile equations with hyperbolic rest points and λ in the region of consistent splitting.

4.6. Proof of Theorem 1

Thanks to Lemma 4.4, it suffices to rule out unstable eigenvalues in a uniformly bounded domain. In view of Lemma 4.11, systems (15) satisfy the hypotheses of Proposition 2.4 in [34]. Therefore for any $\epsilon \in \mathcal{V}$ and in an Ω-neighborhood of some λ, the local Evans functions $D^\epsilon(\lambda)$ converge uniformly to $D^0(\lambda)$ in a (possibly smaller) neighborhood of λ as $\epsilon \to 0^+$, with rate

$$\big|D^\epsilon - D^0\big| \leqslant C\eta(\epsilon) = \mathcal{O}\big(\epsilon + \big|\delta(\epsilon)\big|\big).$$

Since on $\{\operatorname{Re}\lambda \geqslant 0\}$, $D^0(\lambda)$ does not vanish except at $\lambda = 0$ (Corollary 4.3); by analyticity and uniform convergence, the same holds for each D^ϵ for all ϵ near zero. This shows that there are no isolated eigenvalues with finite multiplicity in $\{\operatorname{Re}\lambda \geqslant 0\}$ except $\lambda = 0$. To show that $\lambda = 0$ is a simple eigenvalue, note that by analyticity and uniform convergence of the Evans functions we also have convergence of its derivatives,

$$(d/d\lambda)D^\epsilon(0) \to (d/d\lambda)D^0(0) \neq 0,$$

as $\epsilon \to 0^+$ (non-vanishing by Lemma 4.2), showing that $(d/d\lambda)D^\epsilon(0) \neq 0$ for ϵ sufficiently small. This yields the result.

5. Linear stability and semigroup estimates

In this section we study the linearized evolution equations (13) for the perturbation. Our goal is to prove that system (22) generates a C_0-semigroup in $H^1 \times H^1$, whose infinitesimal generator is spectrally stable if ϵ is sufficiently small. This will yield Theorem 2. From now on we assume that $\epsilon \geqslant 0$ is fixed and we drop the dependence on ϵ for notational convenience. Therefore, from now on, $(\bar{u}, \bar{c})$ will denote the solution to (7) with velocity θ depending on ϵ.

5.1. Semigroups for the uncoupled linearized equations

Recall that a C_0-semigroup in a Banach space X is a family of operators $\{\mathcal{S}(t)\}_{t\geqslant 0}$ satisfying

$$\mathcal{S}(t) \text{ is a bounded linear operator for all } t \geqslant 0, \tag{68}$$

$$\mathcal{S}(0) = I, \tag{69}$$

$$\mathcal{S}(t_1 + t_2) = \mathcal{S}(t_1)\mathcal{S}(t_2), \quad \text{for all } t_1, t_2 \geqslant 0, \tag{70}$$

$$\lim_{t\to 0^+} \mathcal{S}(t)x = x, \quad \text{for all } x \in X. \tag{71}$$

We recall the definition of *growth bound* of a C_0-semigroup,

$$\omega_0 := \inf\Big\{\omega \in \mathbb{R}\colon \lim_{t\to+\infty} e^{-\omega t}\big\|\mathcal{S}(t)\big\| = 0\Big\}. \tag{72}$$

A C_0-semigroup is uniformly (exponentially) stable whenever $\omega_0 < 0$. If $\mathcal{A}$ denotes the infinitesimal generator of a C_0-semigroup $\mathcal{S}(t)$, its *spectral bound* is defined as

$$s(\mathcal{A}) := \sup\big\{\operatorname{Re}\lambda\colon \lambda \in \sigma(\mathcal{A})\big\}, \tag{73}$$

where, of course, $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. Since the spectral mapping theorem

$$\sigma\big(\mathcal{S}(t)\big)\backslash\{0\} = e^{t\sigma(\mathcal{A})}, \quad t \geqslant 0, \tag{74}$$

is not in general true for C_0-semigroups (see [9]), for stability purposes we rely on the Gearhart–Prüss theorem [18,35], which restricts our attention to semigroups on Hilbert spaces (see also [7,9]) and states that any C_0-semigroup $\{\mathcal{S}(t)\}_{t\geqslant 0}$ on a Hilbert space H with infinitesimal generator $\mathcal{A}$ is uniformly exponentially stable if and only if $s(\mathcal{A}) < 0$ and the resolvent satisfies

$$\sup_{\operatorname{Re}\lambda>0} \big\|(\mathcal{A}-\lambda)^{-1}\big\| < +\infty. \tag{75}$$

(See, e.g., Theorem 5.1.11 in [9].)

In order to understand the dynamics of the solutions to (13), we now consider the linearized equations in the absence of the coupling terms $-(\tau'(\bar{c})c)_x$ and $-\epsilon u_x$. Observe, however, that the waves $(\bar{u}, \bar{c})$ remain the solutions to (7) for a certain fixed $\epsilon \geqslant 0$.

5.1.1. The linearized calcium diffusion operator

We start by studying some properties of the linearized calcium diffusion operator in the absence of elastic contraction effects.

Lemma 5.1. *Under assumptions* (2), (4)–(6) *and for each* $\theta > 0$, *the densely defined linear operator*

$$Lc := Dc_{xx} - \theta c_x + R'(\bar{c})c, \qquad \mathcal{D}(L) = H^2 \tag{76}$$

is the infinitesimal generator of a C_0*-semigroup of quasi-contractions in* L^2. *According to custom, we denote this semigroup as* $\{e^{Lt}\}_{t \geqslant 0}$.

Proof. This result is classical (see [32,9]). Here, the linear calcium diffusion operator (76) satisfies the resolvent estimate

$$\left\|(L-\lambda)^{-1}\right\| \leqslant \frac{1}{\operatorname{Re}\lambda - \omega},$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, where $\omega := \max_{c \in [0,1]}\{|R'(c)|\} > 0$, which can be obtained by a straightforward integration by parts. The conclusion follows from the generalized Hille–Yosida theorem [32]. Moreover, the semigroup is quasi-contractive [9],

$$\left\|e^{Lt}\right\| \leqslant e^{\omega t}, \quad t \geqslant 0. \qquad \square \tag{77}$$

Remark 5.2. Here $\bar{c}$ denotes the calcium wave solution to the coupled equations (7) for some $\epsilon \geqslant 0$, and θ depends on ϵ as well. Therefore, $\bar{c}_x$ is not an eigenfunction of the operator (76), nor $\lambda = 0$ is its associated eigenvalue. (In other words, L is not the operator defined in (39).) As a consequence, the Hille–Yosida theorem yields only quasi-contractiveness. Note, however, that the growth rate ω does not depend on ϵ.

5.1.2. The linearized homogeneous elastic equation

We now look at the homogeneous linear elastic equation in the absence of contraction effects due to the calcium concentration. Thus, consider the linear equation

$$\mu u_{xxt} + \mu\theta u_{xxx} + u_{xx} - su = 0. \tag{78}$$

Since we are interested in L^2 solutions to (78), we can solve it by Fourier transform. This leads to

$$-\mu k^2 \hat{u}_t - i\mu\theta k^3 \hat{u} - k^2 \hat{u} - s\hat{u} = 0. \tag{79}$$

Consider an initial condition $u_0 \in L^2$. Then solving for $\hat{u}$ we get

$$\hat{u} = e^{-a(k)t}\hat{u}_0,$$

where

$$a(k) := \frac{1}{\mu} + \frac{s}{\mu k^2} + i\theta k, \quad k \neq 0. \tag{80}$$

By inverse Fourier transform we define the solution operator associated to Eq. (78) as

$$\tilde{\mathcal{S}}(t)u := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} e^{-a(k)t} \hat{u}(k)\, dk, \quad t \geqslant 0, \tag{81}$$

for each $u \in L^2$. This definition is a restatement of $(\tilde{\mathcal{S}}(t)u)\widehat{\ }(k) = e^{-a(k)t}\hat{u}$. Observe that one may consider initial conditions $u_0 \in H^j$, $j = 0, 1, 2$, and define the operators $\tilde{\mathcal{S}}(t) : H^j \to H^j$, on each space H^j, with $j = 0, 1, 2$. We are going to show not only that the operators are well defined on each H^j, but that they constitute stable C_0-semigroups.

Proposition 5.3. *The solution operators $\tilde{\mathcal{S}}(t) : H^j \to H^j$, for $t \geqslant 0$ are well defined on each space H^j, $j = 0, 1, 2$, and constitute exponentially stable C_0-semigroups $\{\tilde{\mathcal{S}}(t)\}_{t\geqslant 0}$ on each H^j, $j = 0, 1, 2$.*

Proof. To show that the operators are well defined on each H^j, note that, by Plancherel theorem, we have for each $u \in H^j$,

$$\big\|\tilde{\mathcal{S}}(t)u\big\|_{L^2}^2 = \big\|e^{-a(k)t}\hat{u}\big\|_{L^2}^2 = \int_{\mathbb{R}} e^{-2t/\mu}e^{-2ts/(\mu k^2)}\big|\hat{u}(k)\big|^2\,dk \leqslant e^{-2t/\mu}\|u\|_{L^2}^2,$$

in view that $\mu > 0$, $s > 0$. Moreover we have, for $j = 0, 1, 2$,

$$\begin{aligned}\big\|\partial_x^j\big(\tilde{\mathcal{S}}(t)u\big)\big\|_{L^2}^2 = \big\|(ik)^j e^{-a(k)t}\hat{u}\big\|_{L^2}^2 &= \int_{\mathbb{R}} e^{-2t/\mu}e^{-2ts/(\mu k^2)}k^{2j}\big|\hat{u}(k)\big|^2\,dk \\ &\leqslant e^{-2t/\mu}\big\|(ik)^j\hat{u}\big\|_{L^2}^2 \leqslant e^{-2t/\mu}\big\|\partial_x^j u\big\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\big\|\tilde{\mathcal{S}}(t)u\big\|_{H^j} \leqslant e^{-t/\mu}\|\hat{u}\|_{H^j} = Ce^{-t/\mu}\|u\|_{H^j} < \infty, \tag{82}$$

for all $t \geqslant 0$, with some $C > 0$, and with $C \equiv 1$ for $j = 0$. (The semigroup $\tilde{\mathcal{S}}(t)$ is contractive in L^2.) This shows not only (68), but also that each $\tilde{\mathcal{S}}(t)$ is uniformly exponentially stable in each H^j. Likewise, note that $\tilde{\mathcal{S}}(0) = I$ is simply a restatement of the Fourier inversion theorem. To check (70), one writes

$$\tilde{\mathcal{S}}(t_2)\tilde{\mathcal{S}}(t_1)u = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}e^{-a(k)t_2}\big(\tilde{\mathcal{S}}(t_1)u\big)\widehat{\ }(k)\,dk.$$

Whence, by the inversion formula with fixed t_1, we obtain

$$\big(\tilde{\mathcal{S}}(t_1)u\big)\widehat{\ }(k) = e^{-a(k)t_1}\hat{u}(k),$$

yielding

$$\tilde{\mathcal{S}}(t_2)\tilde{\mathcal{S}}(t_1)u = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}e^{-a(k)(t_2+t_1)}\hat{u}(k)\,dk = \tilde{\mathcal{S}}(t_2+t_1)u.$$

By a well-known result,[2] once (69) and (70) hold, then in order to prove (71) it suffices to show

$$\operatorname*{w-lim}_{t\to 0^+}\tilde{\mathcal{S}}(t)u = u, \quad \text{for all } u \in H^j, \tag{83}$$

i.e., $\langle\tilde{\mathcal{S}}(t)u, \varphi\rangle_{H^j} \to \langle u, \varphi\rangle_{H^j}$, as $t \to 0^+$ for all $\varphi \in H^j$. By Parseval relation,

$$\big\langle\tilde{\mathcal{S}}(t)u, \varphi\big\rangle_{L^2} = \big\langle e^{-a(k)t}\hat{u}, \hat{\varphi}\big\rangle_{L^2} = \int_{\mathbb{R}} e^{+i\theta kt}e^{-\mu^{-1}(1+s/k^2)t}\hat{u}^*\hat{\varphi}\,dk,$$

[2] See Yosida [42, p. 233].

and, furthermore, for each $j = 0, 1, 2$,

$$\begin{aligned}\left\langle \partial_x^j\big(\tilde{\mathcal{S}}(t)u\big), \partial_x^j\varphi\right\rangle_{L^2} &= \left\langle (ik)^j e^{-a(k)t}\hat{u}, (ik)^j\hat{\varphi}\right\rangle_{L^2}\\ &= \int_{\mathbb{R}} e^{+i\theta kt} e^{-\mu^{-1}(1+s/k^2)t}(-1)^j k^{2j}\hat{u}^*\hat{\varphi}\,dk.\end{aligned}$$

Assuming $u, \varphi \in C_0^\infty$ we can take the limit inside the integral to find

$$\lim_{t\to 0^+}\left\langle \partial_x^j\big(\tilde{\mathcal{S}}(t)u\big), \partial_x^j(\varphi)\right\rangle_{L^2} = \int_{\mathbb{R}} (-1)^j k^{2j}\hat{u}^*\hat{\varphi}\,dk = \left\langle (ik)^j\hat{u}, (ik)^j\hat{\varphi}\right\rangle_{L^2} = \left\langle \partial_x^j u, \partial_x^j\varphi\right\rangle_{L^2}.$$

This shows that for $u, \varphi \in C_0^\infty$ one has

$$\lim_{t\to 0^+}\left\langle \tilde{\mathcal{S}}(t)u, \varphi\right\rangle_{H^j} = \langle u, \varphi\rangle_{H^j}, \quad j = 0, 1, 2.$$

By density, in the case $u, \varphi \in H^j$, there exist sequences u_n, φ_m in C_0^∞ converging in H^j to u and φ, respectively. Therefore, taking into account that each $\tilde{\mathcal{S}}(t)$ is uniformly stable (and bounded), we get

$$\begin{aligned}\left|\left\langle \tilde{\mathcal{S}}(t)u - u, \varphi\right\rangle_{H^j}\right| &= \left|\left\langle \tilde{\mathcal{S}}(t)(u - u_n) + \tilde{\mathcal{S}}(t)u_n - u_n - (u - u_n), (\varphi - \varphi_m) + \varphi_m\right\rangle_{H^j}\right|\\ &\leqslant 2C\|u - u_n\|_{H^j}\|\varphi - \varphi_m\|_{H^j} + 2C\|u - u_n\|_{H^j}\|\varphi_m\|_{H^j}\\ &\quad + 2\|u_n\|_{H^j}\|\varphi - \varphi_m\|_{H^j} + \left|\left\langle \tilde{\mathcal{S}}(t)u_n - u_n, \varphi_m\right\rangle_{H^j}\right|.\end{aligned}$$

In view that $\langle \tilde{\mathcal{S}}(t)u_n - u_n, \varphi_m\rangle_{H^j} \to 0$ as $t \to 0^+$ for all n, m, then for given $\varepsilon > 0$, we choose n and m sufficiently large such that

$$\left|\left\langle \tilde{\mathcal{S}}(t)u - u, \varphi\right\rangle_{H^j}\right| \leqslant \varepsilon,$$

for all $t \sim 0^+$, proving (83). □

Remark 5.4. Although Eq. (78) is not in standard evolution form, in principle we can define the infinitesimal generator of the semigroup $\{\tilde{\mathcal{S}}(t)\}_{t\geqslant 0}$ in each H^j, $j = 0, 1, 2$, by

$$\tilde{\mathcal{A}}u := \lim_{t\to 0^+} t^{-1}\big(\tilde{\mathcal{S}}(t)u - u\big) \quad \text{in } H^j,\ u \in \mathcal{D}(\tilde{\mathcal{A}}),$$

with domain

$$\mathcal{D}(\tilde{\mathcal{A}}) := \left\{u \in H^j\colon \lim_{t\to 0^+} t^{-1}\big(\tilde{\mathcal{S}}(t)u - u\big) \text{ exists in } H^j\right\} \subset H^j,$$

and (78) is equivalent to an evolution equation of form $u_t = \tilde{\mathcal{A}}u$. Moreover, $\mathcal{D}(\tilde{\mathcal{A}})$ is dense in H^j, $\tilde{\mathcal{A}}$ is closed, and the mapping $t \mapsto \tilde{\mathcal{S}}(t)u$ is C^1 in $t > 0$ for $u \in \mathcal{D}(\tilde{\mathcal{A}})$ (see [32]).

Consistently with semigroup theory [32], we define, for each $u_0 \in L^2$, the *mild* solution to (78) as the function $u = \tilde{\mathcal{S}}(t)u_0 \in C([0, T]; L^2)$ for each $T > 0$. By uniform exponential stability of the semigroup this solution is global, $u \in C([0, +\infty); L^2)$. Proposition 5.3 establishes the existence of global mild solutions to (78) by means of formula (81). Our aim is now to prove that mild solutions are strong solutions (in the sense that $u(\cdot, t) \in H^3$ for $t \geqslant 0$, and $u_t(\cdot, t) \in H^2$ for $t > 0$), provided

$u_0 \in H^3 \cap L^1$. This is the content of Lemma 5.6 below. First we establish the basic decay properties of the solution.

Lemma 5.5. *Let $m \in \mathbb{N}$, $u_0 \in H^m \cap L^1$ and*

$$\phi_j(k,t) = k^j e^{-a(k)t} \hat{u}_0(k).$$

Then:

(a) $\phi_j(t) \in L^2$ *for* $t \geqslant 0$, $j = 0, \dots, m$.
(b) $\phi_j(t) \in L^1$ *for* $t \geqslant 0$, $j = 0, \dots, m-1$.
(c) $\phi_j(t) \in L^1 \cap L^2$ *for* $t > 0$, $j = -2, -1$.
(d) *For each* $m \geqslant 2$, $k^j \partial_t \phi_0 \in L^2 \cap L^1$ *for* $t > 0$, $j = 0, \dots, m-1$.

Proof. (a) For $j \in \{0, \dots, m\}$ we have $|\phi_j(k,t)| \leqslant k^j |\hat{u}_0(k)|$; thus, $\phi_j(t) \in L^2$.

(b) For $j \in \{0, \dots, m-1\}$ and $|k| \geqslant 1$, $|\phi_j(k,t)| \leqslant |k^{j+1} \hat{u}_0(k)|/|k|$, while for $|k| \leqslant 1$, $|\phi_j(k,t)| \leqslant \|\hat{u}_0\|_{L^\infty} \leqslant \|u_0\|_{L^1}$.

(c) For $|k| \geqslant 1$, $|\phi_j(k,t)| \leqslant |\hat{u}_0(k)|/|k|$, $j = -2, -1$. For $|k| \leqslant 1$,

$$\left|\phi_{-1}(k,t)\right| \leqslant \frac{1}{\sqrt{2e}} \frac{\|u_0\|_{L^1}}{\sqrt{t}}, \qquad \left|\phi_{-2}(k,t)\right| \leqslant \frac{1}{et} \|u_0\|_{L^1}.$$

(d) Since

$$k^j \partial_t \phi_0 = -a(k) k^j \phi_0(k,t) = -\big(1/\mu + i\theta k + s/\big(\mu k^2\big)\big)\phi_j(k,t),$$

then (d) follows from (a), (b) and (c). □

We can now prove that mild solutions are strong solutions.

Lemma 5.6. *If $u_0 \in H^3 \cap L^1$, then $u(x,t)$ defined by* (81) *satisfies $u(\cdot,t) \in H^3$ for $t \geqslant 0$, $u_t(\cdot,t) \in H^2$ for $t > 0$, and u is a strong solution of* (78).

Proof. In Lemma 5.5 we proved that $\partial_t \phi_0 = \partial_t \hat{u} \in L^1$ for $t > 0$. Hence, we can take derivative with respect to t in the Fourier representation of u to get $\hat{u}_t = (u_t)\widehat{\ }$. We apply Lemma 5.5 to conclude that $u(t) \in H^3$, $u_t \in H^2$ and $(\partial^j u/\partial x^j)\widehat{\ } = (ik)^j \hat{u}$, for all $j = 1, 2, 3$. By regularity and its definition in terms of the Fourier transform, it is clear that $\hat{u}$ satisfies (79), and that u satisfies (78) a.e. This establishes the equivalence of solutions to (78) with initial condition $u(x,0) = u_0$ and shows that mild solutions are strong solutions. □

Remark 5.7. Under the assumption $u_0 \in H^4 \cap L^1$, $u(x,t)$ is a smooth solution to (78), that is, $u \in C^3$, $u_t \in C^2$. Moreover $\partial^j u/\partial x^j$ for $j = 0, 1, 2, 3$ and $\partial^j u_t/\partial x^j$ for $j = 0, 1, 2$ decay to 0 as $|x| \to +\infty$ by the Riemann–Lebesgue lemma.

Observe that if $u(x,t)$ is a regular solution to (78), then integrating the equation in $x \in \mathbb{R}$ we find that the perturbation has zero-mass,

$$\int_{\mathbb{R}} u(x,t)\, dx = 0,$$

for all $t > 0$ (in view that $s \neq 0$). The latter is a necessary condition for its antiderivative

$$U(x,t) := \int_{-\infty}^{x} u(y,t)\,dy$$

to belong to L^2. Integrating (78) once more, we notice that the antiderivative must have zero-mass as well,

$$\int_{\mathbb{R}} U(x,t)\,dx = 0,$$

for each $t > 0$. This suggests that the dynamics of the solutions to (78) takes place in the subset of L^2 consisting of functions with "double zero-mass" (meaning that both the function and its antiderivative have mean zero) regardless of the initial condition. This is the content of the following

Proposition 5.8. *Let $u_0 \in H^3 \cap L^1$ and $u = \tilde{S}(t)u_0$. Then, for all $t > 0$,*

$$\int_{\mathbb{R}} u(x,t)\,dx = 0,$$

$$U(x,t) := \int_{-\infty}^{x} u(y,t)\,dy \in L^2,$$

and

$$\int_{\mathbb{R}} U(x,t)\,dx = 0.$$

Proof. Let

$$\hat{U}(k,t) = -\frac{i}{k}\hat{u}(k,t) = -\frac{i}{k}e^{-a(k)t}\hat{u}_0(k);$$

then $\hat{U}(k,t) \in L^2$ for all $t > 0$ as it was proved in Lemma 5.5. Thus, by the Fourier inverse theorem

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx}\hat{U}(k,t)\,dk \in L^2,$$

for all $t > 0$. We also have from (c) of Lemma 5.5, that $\hat{U}(k,t) \in L^1$ for $t > 0$. It follows that $U_x = u$ and by the Riemann–Lebesgue lemma, that $U(x,t)$ is continuous and it decays to zero as $|x| \to +\infty$. Therefore

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy \quad \text{and} \quad 0 = U(+\infty,t) = \int_{\mathbb{R}} u(x,t)\,dx.$$

The proof that U also has zero-mass is similar: let $\hat{w}(k,t) = -(i/k)\hat{U}(k,t) = -(1/k^2)\hat{u}(k,t)$. Hence $w = (\hat{w})^{\vee}$ is in L^2, it is continuous and it decays to zero as $|x| \to +\infty$. Thus,

$$w(x,t) = \int_{-\infty}^{x} U(y,t)\,dy \quad \text{and} \quad 0 = w(+\infty,t) = \int_{\mathbb{R}} U(x,t)\,dx,$$

for all $t > 0$. $\square$

In view of last observations, we look at the subset of L^2 functions u satisfying

$$\int_{\mathbb{R}} u(x)\,dx = 0 \tag{84}$$

and

$$\int_{\mathbb{R}} \int_{-\infty}^{x} u(y)\,dy\,dx = 0, \tag{85}$$

together with the conditions

$$U(x) := \int_{-\infty}^{x} u(y)\,dy \in L^2, \tag{86}$$

$$V(x) := \int_{-\infty}^{x} \int_{-\infty}^{y} u(\zeta)\,d\zeta\,dy \in L^2. \tag{87}$$

Let us denote this set as

$$\mathcal{U} := \big\{ u \in L^2\colon\ u \text{ satisfies (84), (85), (86) and (87)} \big\} \subset L^2.$$

As a by-product of Proposition 5.8 we obtain the following

Corollary 5.9. *$\mathcal{U} \cap L^1 \cap H^j$ is dense in H^j, for each $j = 0, 1, 2$.*

Proof. Let $u \in H^j$. By density of C_0^∞ in H^j, there exists $u_n \in C_0^\infty$ such that $u_n \to u$ in H^j as $n \to +\infty$. Let us denote $\psi_n(t) := \tilde{\mathcal{S}}(t)u_n$ for each $t > 0$. Since each u_n is smooth, $u_n \in C_0^\infty \subset H^3$; then by Lemma 5.5(c), Proposition 5.8 and Lemma 5.18, $\psi_n(t)$ has double zero-mass, $\psi_n(t) \in H^2$ for each $t > 0$, and it is integrable for each $t > 0$, i.e., $\psi_n(t) \in \mathcal{U} \cap L^1 \cap H^2 \subset \mathcal{U} \cap L^1 \cap H^j$ for each $j = 0, 1, 2$. By convergence and semigroup properties we have

$$\big\| \psi_n(t) - u \big\|_{H^j} \leqslant \big\| \tilde{\mathcal{S}}(t)u_n - u_n \big\|_{H^j} + \|u_n - u\|_{H^j} \leqslant \varepsilon,$$

with ε arbitrarily small, for each n sufficiently large and when $t \to 0^+$. This proves the result. $\square$

Remark 5.10. Note that this dense inclusion may be also verified by pointing out that the zero-mass conditions (84)–(87) imply that the Fourier transform of $u \in \mathcal{U} \cap L^1 \cap H^j$ has a double zero at $k = 0$. Notably, our proof makes use of the constructed semigroup.

As a consequence of the restrictions (86) and (87) we also have the following

Corollary 5.11. *If $u \in L^1 \cap \mathcal{U}$ then*

$$\int_{\mathbb{R}} k^{-2} \left|\hat{u}(k)\right|^2 dk < +\infty \tag{88}$$

and

$$\int_{\mathbb{R}} k^{-4} \left|\hat{u}(k)\right|^2 dk < +\infty. \tag{89}$$

Proof. $u \in L^1$ implies that

$$U(x) := \int_{-\infty}^{x} u(y)\, dy$$

satisfies $U_x = u$ at every Lebesgue point of u, hence a.e. in $x \in \mathbb{R}$. Moreover, $u \in \mathcal{U}$ implies $U \in L^2$. Therefore $ik\hat{U} = \hat{u}$ and

$$\int_{\mathbb{R}} k^{-2} \left|\hat{u}(k)\right|^2 dk = \int_{\mathbb{R}} \left|\hat{U}(k)\right|^2 < +\infty.$$

Likewise, there exists $V \in L^2$ such that $V_{xx} = u$ a.e., and consequently $-k^2\hat{V} = \hat{u}$, yielding

$$\int_{\mathbb{R}} k^{-4} \left|\hat{u}(k)\right|^2 dk = \int_{\mathbb{R}} \left|\hat{V}(k)\right|^2 < +\infty,$$

as claimed. □

5.2. The coupled linear system

We now look at the coupled linear system for the perturbations (13), which, recast in terms of the deformation gradient $v = u_x$, has the form (22) (written here again for convenience of the reader),

$$\begin{aligned} \mu\theta v_{xx} + \mu v_{xt} + v_x - s \int_{-\infty}^{x} v(y,t)\, dy - (\tau'(\bar{c})c)_x = 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c = 0. \end{aligned} \tag{22}$$

Remark 5.12. Consider the following densely defined (by Corollary (5.9)) operator in H^1,

$$\mathcal{J} : \mathcal{U} \cap L^1 \cap H^1 \subset H^1 \to H^1, \tag{90}$$

$$\mathcal{J} v := \int_{-\infty}^{x} \int_{-\infty}^{y} v(\zeta)\, d\zeta\, dy. \tag{91}$$

If we study perturbations in the deformation gradient variable v with the double mean zero property (or equivalently, with zero-mass perturbations in the elastic variable u), one may write alternatively the resulting linear system (22) for (v, c) as

$$\begin{aligned} \mu\theta v_x + \mu v_t + v - s\mathcal{J}v - \tau'(\bar{c})c &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon v - R'(\bar{c})c &= 0, \end{aligned} \tag{92}$$

and look for solutions with v in the domain of the operator $\mathcal{J}$.

Motivated by semigroup theory [32,9] we define mild solutions to system (22) (or (92)) in terms of the variations of constants formula for the Fourier transform.

Definition 5.13. We say $(v, c) \in C([0, +\infty); H^1 \times H^1)$ is a global mild solution to (22) (or equivalently, to (92)) with initial condition $(v_0, c_0) \in H^1 \times H^1$ provided that

$$\hat{v}(k,t) = e^{-a(k)t}\hat{v}_0(k) + \int_0^t e^{-a(k)(t-\sigma)}\big(\tau'(\bar{c})c\big)\widehat{\ }(k,\sigma)\,d\sigma, \tag{93}$$

$$\hat{c}(k,t) = e^{-(Dk^2+ik\theta)t}\hat{c}_0(k) + \int_0^t e^{-(Dk^2+ik\theta)(t-\sigma)}\big(\epsilon\hat{v}(k,\sigma) + \big(R'(\bar{c})c\big)\widehat{\ }(k,\sigma)\big)\,d\sigma \tag{94}$$

hold for all $t \in [0, +\infty)$, where, as usual, $a(k) = 1/\mu + i\theta k + s/(\mu k^2)$, for $k \neq 0$. We say that a mild solution is a strong solution provided that $(v, c)(\cdot, t) \in H^2$ for each $t \geqslant 0$ and $(v, c)_t(\cdot, t) \in H^1$ for each $t > 0$.

5.2.1. Existence and uniqueness of mild solutions

The following is the main result of this section.

Lemma 5.14. *Let $(v_0, c_0) \in H^1 \times H^1$. Then there exists a unique global mild solution $(v, c) \in C([0, +\infty); H^1 \times H^1)$ to system (22) with initial condition $(v, c)(0) = (v_0, c_0)$.*

Proof. Let $T > 0$ and define the space of curves

$$X := \big\{c \in C\big([0, T]; H^1\big)\colon\ c(0) = c_0\big\}.$$

Consider a curve $c \in X$ and let v^c be the solution to

$$\begin{aligned} \mu\theta v^c_x + \mu v^c_t + v^c - s\mathcal{J}v^c &= \tau'(\bar{c})c, \\ v^c(0) &= v_0. \end{aligned} \tag{95}$$

For each $c(t) \in X$, the solution v^c is completely determined for the variation of constants formula

$$\hat{v}^c(k,t) = e^{-a(k)t}\hat{v}_0 + \int_0^t e^{-a(k)(t-\sigma)}\big(\tau'(\bar{c})c\big)\widehat{\ }(k,\sigma)\,d\sigma. \tag{96}$$

Observe that since $c(t) \in X$ then

$$\left\|\tau'(\bar{c})c(t)\right\|_{L^2}^2 \leqslant M\left\|c(t)\right\|_{H^1}^2,$$

$$\left\|\left(\tau'(\bar{c})c\right)_x(t)\right\|_{L^2}^2 \leqslant M\left\|c(t)\right\|_{H^1}^2,$$

with $M := \max_{c\in[0,1]}\{|\tau'(c)|^2, |\tau''(c)|^2 C_0^2\} > 0$, where $C_0 > 0$ is a uniform constant for which $|\bar{c}| \leqslant C_0 e^{-|x|/C_1}$ by (11). Thus, $\tau'(\bar{c})c(t) \in H^1$ for each $t \in [0,T]$, and consequently $v^c(t) \in H^1$, for each $t \in [0,T]$, $c(t) \in X$. Moreover we have the estimate

$$\left\|v^c(t)\right\|_{H^1} \leqslant e^{-t/\mu}\|v_0\|_{H^1} + \mu M \max_{0\leqslant\sigma\leqslant t}\left\|c(\sigma)\right\|_{H^1}. \tag{97}$$

Now with $c(t) \in X$, we then define a map $\Phi : X \to X$ by

$$\xi =: \Phi(c),$$

where ξ is the solution to

$$\xi_t + \theta\xi_x - D\xi_{xx} = \epsilon v^c + R'(\bar{c})c,$$

$$\xi(x,0) = c_0.$$

The last linear equation for ξ can be solved by means of the Fourier transform to obtain, by the variation of constants formula,

$$\hat{\xi}(k,t) = e^{-(Dk^2+ik\theta)t}\hat{c}_0(k) + \int_0^t e^{-(Dk^2+ik\theta)(t-\sigma)}\left(\epsilon\hat{v}^c(k,\sigma) + \left(R'(\bar{c})c\right)\widehat{\ }(k,\sigma)\right)d\sigma. \tag{98}$$

Clearly, since $c_0 \in H^1$, $v^c(t) \in H^1$ and $(R'(\bar{c})c)(t) \in H^1$ (same argument like for τ, as $|R'|$, $|R''|$ bounded in $c \in [0,1]$), we have that $\xi \in X$.

The next step is to verify that Φ is a contraction for $T > 0$ small enough. Indeed, if $c_1, c_2 \in X$, let us call v_1 and v_2 the corresponding solutions to (95) given by (96). From representation formula (98) and estimate (97), and since ξ_1 and ξ_2 have the same initial condition, we have for $t \in [0,T]$ that

$$\begin{aligned}\left\|\xi_1(t) - \xi_2(t)\right\|_{H^1} &\leqslant \epsilon\int_0^t \left\|v_1(\sigma) - v_2(\sigma)\right\|_{H^1}d\sigma + M\int_0^t\left\|c_1(\sigma) - c_2(\sigma)\right\|_{H^1}d\sigma \\ &\leqslant (1+\epsilon\mu)MT\max_{0\leqslant\sigma\leqslant T}\left\|c_1(\sigma) - c_2(\sigma)\right\|_{H^1}.\end{aligned}$$

It follows that Φ is a contraction for $T \leqslant T_1 := \frac{1}{2}((1+\epsilon\mu)M)^{-1}$. Therefore Φ has a unique fixed point $c \in X$ which defines a solution to (96) and (98). By construction, (v,c) solves system (22) in $t \in [0,T_1]$. This solution can be continued to the time interval $[jT_1,(j+1)T_1]$, $j \in \mathbb{N}$, hence it is defined for all $t \geqslant 0$. Moreover, solutions are unique as Φ is a contraction on each interval of length T_1 and they are clearly continuous mappings from $[0,T]$ to $H^1 \times H^1$ for each $T > 0$. This shows global existence and uniqueness of mild solutions $(v,c) \in C([0,+\infty); H^1 \times H^1)$. □

Remark 5.15. Notice that formulas (96) and (98) provide, by Duhamel principle, a representation of the solutions $(v, c) \in C([0, +\infty); H^1 \times H^1)$ of Lemma 5.14, given by (93) and (94) for all $t \in [0, +\infty)$. Alternatively we may also express the calcium concentration solution $c(t)$ as

$$c(t) = e^{tL} c_0 + \epsilon \int_0^t e^{(t-\zeta)L} v(\zeta)\, d\zeta, \tag{99}$$

where $\{e^{tL}\}_{t \geqslant 0}$ is the C_0-semigroup in L^2 (but not necessarily in H^1) of Lemma 5.1.

5.2.2. Semigroup properties

For each $(v_0, c_0) \in H^1 \times H^1$ we denote the solution constructed in Lemma 5.14 as

$$\mathcal{S}(t)(v_0, c_0) := (v, c)(t), \tag{100}$$

for each $t \geqslant 0$.

Lemma 5.16. *The solution operator* (100) *constitutes as C_0-semigroup $\{\mathcal{S}(t)\}_{t \geqslant 0}$ in $H^1 \times H^1$.*

Proof. Clearly, each $S(t)$ is a linear bounded operator, $\mathcal{S}(t) : H^1 \times H^1 \to H^1 \times H^1$, with $\mathcal{S}(0) = I$, showing (68) and (69). To verify the semigroup property (70), we observe that the equations are autonomous and that solutions are unique. Thus, since $\mathcal{S}(t)\mathcal{S}(t_1)(v_0, c_0)$ solves (22) for each $t \geqslant t_1$, it coincides with the solution $\mathcal{S}(t + t_1)(v_0, c_0)$. This shows (70). To prove continuity (71), take $t \in [0, 1]$. Then by representation formula (93)

$$\|v(t) - v_0\|_{H^1} \leqslant \|(e^{-a(k)t} - 1)\hat{v}_0\|_{H^1} + Mt \max_{0 \leqslant \zeta \leqslant t} \|c(\zeta)\|_{H^1}.$$

The second term clearly goes to zero as $t \to 0^+$. The first term also decays to zero,

$$\|(e^{-a(k)t} - 1)\hat{v}_0\|_{H^1} = \|(\tilde{\mathcal{S}}(t) - I)v_0\|_{H^1} \to 0,$$

because $\tilde{\mathcal{S}}(t)$ is a C_0-semigroup in H^1 by Proposition 5.3. This shows

$$\lim_{t \to 0^+} \|v(t) - v_0\|_{H^1} = 0.$$

From formula (94),

$$\|c(t) - c_0\|_{H^1} \leqslant \|(1 - e^{-(Dk^2 + ik\theta)t})\hat{c}_0\|_{H^1} + (\epsilon + M)t \max_{0 \leqslant \zeta \leqslant t} (\|v(\zeta)\|_{H^1} + \|c(\zeta)\|_{H^1}).$$

The second term clearly goes to zero as $t \to 0^+$. To show that the first term tends to zero as well, notice that in view of $|1 - e^{-(Dk^2 + ik\theta)t}| \to 0$, as $t \to 0^+$ for each $k \in \mathbb{R}$, and since $|\hat{c}_0(k)|^2, k^2|\hat{c}_0(k)| \in L^1$, because $c_0 \in H^1$, it follows from the dominated convergence theorem that

$$\lim_{t \to 0^+} \|(1 - e^{-(Dk^2 + ik\theta)t})\hat{c}_0\|_{L^2} = 0,$$

$$\lim_{t \to 0^+} \|(1 - e^{-(Dk^2 + ik\theta)t})(ik)\hat{c}_0\|_{L^2} = 0.$$

This shows

$$\lim_{t\to 0^+} \left\| c(t) - c_0 \right\|_{H^1} = 0,$$

and the lemma is proved. □

5.2.3. Regularity and strong solutions

Our goal is now to prove that solutions (93) and (94) give rise to strong solutions of (22). We begin with the basic decay properties of $\hat{v}(k,t)$.

Lemma 5.17. *Let*

$$\psi(k,t) = \int_0^t e^{-a(k)(t-\zeta)} \hat{h}(k,\zeta)\, d\zeta,$$

where $h \in C([0,T]; L^2)$, $h(t) \in H^2 \cap L^1$ *for all* $0 \leqslant t \leqslant T$, *is such that*

$$\left\| h(t) \right\|_{H^2} + \left\| h(t) \right\|_{L^1} \leqslant K_T,$$

for $t \in [0,T]$. *Then*:

(a) $k^j \psi(k,t) \in L^2$ *for* $t \geqslant 0$, $j = 0, 1, 2$.
(b) $k^j \psi(k,t) \in L^1$ *for* $t \geqslant 0$, $j = 0, 1$.
(c) $k^j \psi(k,t) \in L^1 \cap L^2$ *for* $t > 0$, $j = -2, -1$.
(d) $k^j \partial_t \psi(k,t) \in L^2 \cap L^1$ *for* $t > 0$, $j = 0, 1$.

Proof. (a) For $j = 0, 1, 2$ one has

$$\begin{aligned}\left\| k^j \psi(k,t) \right\|_{L^2} &\leqslant \int_0^t \left\| e^{-a(k)(t-\zeta)} k^j \hat{h}(k,t) \right\|_{L^2} d\zeta \\ &\leqslant \int_0^t e^{-a(k)(t-\zeta)} \left\| k^j \hat{h}(k,t) \right\|_{L^2} d\zeta \leqslant \mu K_T.\end{aligned}$$

This shows (a).

(b) For $j = 0, 1$ and $|k| \geqslant 1$ there holds

$$\left| k^j \psi(k,t) \right| = \frac{1}{|k|} \left| k^{j+1} \psi(k,t) \right|,$$

while for $|k| \leqslant 1$, $|\psi(k,t)| \leqslant \mu K_T$.

(c) For $|k| \geqslant 1$, $j = -2, -1$ we have $k^j \psi(k,t) \in L^2 \cap L^1$ for each $t \in [0,T]$. For $|k| \leqslant 1$ one has

$$\left| k^{-1} \psi(k,t) \right| \leqslant \frac{1}{\sqrt{2e}} \int_0^t \frac{d\zeta}{\sqrt{t-\zeta}} \max_{0 \leqslant t \leqslant T} \left\| h(t) \right\|_{L^1} \leqslant \frac{\sqrt{2} K_T}{\sqrt{e}} \sqrt{t},$$

and

$$\left|k^{-2}\psi(k,t)\right| \leqslant \frac{1}{|k|^{1/4}} \int_0^t \frac{e^{-(t-\zeta)/k^2}}{|k|^{7/4}} \left|\hat{h}(k,t)\right| d\zeta \leqslant \frac{CK_T t^{1/8}}{|k|^{1/4}},$$

yielding $k^{-2}\psi(k,t) \in L^2([-1,1]) \cap L^1([-1,1])$. This shows (c).

(d) Finally, since $k^j\psi_t = k^j\hat{h}(k,t) - k^j a(k)\psi(k,t)$, the required properties are obtained from (a), (b) and (c). □

We can now prove that mild solutions are strong solutions.

Lemma 5.18. *If $v_0 \in H^2 \cap L^1$, $c_0 \in H^2$, then $(v,c)(t)$ defined by (93) and (94) satisfy $v(t), c(t) \in H^2$, $v_t \in H^1$ for each $t > 0$ and they are strong solutions to (22).*

Proof. It is clear from (94) that $c(t) \in H^2$. Since $\hat{v}(k,t) = \phi_0(k,t) + \psi(k,t)$ with ϕ_0 solution of the homogeneous elastic equation, Lemmas 5.6 and 5.17 imply that $v(t) \in H^2$, $v_t \in H^1$ for $t > 0$. Moreover, Lemmas 5.5 and 5.17 imply that $\hat{v}_t \in L^1$ for $t > 0$, which yields $\hat{v}_t = (v_t)\widehat{\ }$. We also have $(\partial^j v/\partial x^j)\widehat{\ } = (ik)^j\hat{v}$ for $j = 1, 2$. This shows that solutions to (93) and (94) are strong solutions to (22). □

Remark 5.19. Under the assumption $v_0 \in H^3 \cap L^1$, the solution $v(t) \in C^2$, $v_t \in C^1$ for each $t > 0$ and v, v_x, v_{xx}, v_t and v_{tx} decay to zero as $|x| \to +\infty$. This fact is a consequence of Lemma 5.17.

As in the case of the homogeneous equation, the dynamics of the solutions to the coupled system takes place in the subspace of functions with double zero-mass.

Proposition 5.20. *Let $v(t), c(t)$ be the solutions (93) and (94) with $v_0 \in H^2 \cap L^1$ and $c_0 \in H^2$. Then, for each $t > 0$,*

$$\begin{aligned} &\int_{\mathbb{R}} v(x,t)\,dx = 0, \\ &u(x,t) = \int_{-\infty}^{x} v(y,t)\,dy \in L^2, \quad \textit{and} \\ &\int_{\mathbb{R}} u(x,t)\,dx = 0. \end{aligned} \tag{101}$$

Proof. By Lemma 5.17(c), we have that $-i\hat{v}(k,t)/k \in L^2 \cap L^1$ for $t > 0$; therefore

$$u(x,t) := -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{ikx}}{k} \hat{v}(k,t)\,dk$$

is in L^2, it is continuous, and it decays to zero as $|x| \to +\infty$. Moreover, $u_x = v$, so that $u(x,t) = \int_{-\infty}^{x} v(y,t)\,dy$ and

$$0 = u(+\infty,t) = \int_{\mathbb{R}} v(x,t)\,dx,$$

for $t > 0$. Similarly,

$$U(x,t) := \int_{-\infty}^{x} u(y,t)\,dy$$

decays to zero as $|x| \to +\infty$ and

$$0 = U(+\infty,t) = \int_{\mathbb{R}} u(x,t)\,dx. \qquad \square$$

5.3. The infinitesimal generator $\mathcal{A}$

In view of the results of last section, the constructed C_0-semigroup $\mathcal{S}(t)$ has an infinitesimal generator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H^1 \times H^1 \to H^1 \times H^1, \tag{102}$$

defined by

$$\mathcal{A}(v,c) := \lim_{t \to 0^+} t^{-1}\big(\mathcal{S}(t)(v,c) - (v,c)\big) \quad \text{in } H^1 \times H^1, \tag{103}$$

on the domain

$$\mathcal{D}(\mathcal{A}) := \Big\{(v,c) \in H^1 \times H^1 \colon \lim_{t \to 0^+} t^{-1}\big(\mathcal{S}(t)(u,c) - (u,c)\big) \text{ exists in } H^1 \times H^1\Big\}. \tag{104}$$

By standard semigroup theory [9,32], it is well known that $\mathcal{D}(\mathcal{A})$ is dense in $H^1 \times H^1$, and that $\mathcal{A}$ is a closed, densely defined operator such that the mapping $t \mapsto \mathcal{S}(t)(v,c)$ is of class C^1 in $t > 0$ for all $(v,c) \in \mathcal{D}(\mathcal{A})$, with the property that

$$\frac{d}{dt}\big(\mathcal{S}(t)(v,c)\big) = \mathcal{S}(t)\mathcal{A}(v,c) = \mathcal{A}\mathcal{S}(t)(v,c), \tag{105}$$

for all $(v,c) \in \mathcal{D}(\mathcal{A})$.

Inspection of the linear system (92) for (v,c) yields an explicit expression for the operator $\mathcal{A}$, namely

$$\big(\mathcal{A}(v,c)\big)^\top = \begin{pmatrix} (s/\mu)\mathcal{J} - 1/\mu - \theta\partial_x & \tau'(\bar{c})/\mu \\ \epsilon & D\partial_x^2 - \theta\partial_x + R'(\bar{c}) \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix}. \tag{106}$$

Recall that $\mathcal{J} : L^1 \cap \mathcal{U} \subset H^1 \to H^1$ is the operator densely defined in (91). Observe that the spectral problem $(\mathcal{A} - \lambda)(v,c) = 0$ can be written as the system of equations

$$\begin{aligned} \mu\theta v_x + (\mu\lambda + 1)v - s\mathcal{J}v - \tau'(\bar{c})c &= 0, \\ \lambda c + \theta c_x - Dc_{xx} - \epsilon v - R'(\bar{c})c &= 0, \end{aligned} \tag{107}$$

for eigenfunction $(v,c) \in H^1 \times H^1$.

Since $\mathcal{J}$ maps to H^1 provided that $v \in L^1 \cap H^1$ satisfies the double zero-mass conditions (84) and (85) (i.e., $v \in \mathcal{U} \cap L^1$), the domain of $\mathcal{A}$ must contain sufficiently regular functions with v in $\mathcal{U} \cap L^1$. More precisely we have the following

Lemma 5.21. $(H^2 \cap L^1 \cap \mathcal{U}) \times H^3 \subseteq \mathcal{D}(\mathcal{A})$.

Proof. Let $(v_0, c_0) \in (H^2 \cap L^1 \cap \mathcal{U}) \times H^3$. It suffices to show that both

$$I_1(t) := \left\| t^{-1}\big(c(t) - c_0\big) \right\|_{H^1} \quad \text{and} \quad I_2(t) := \left\| t^{-1}\big(v(t) - v_0\big) \right\|_{H^1}$$

(with $(v(t), c(t)) := \mathcal{S}(t)(v_0, c_0))$ have a limit when $t \to 0^+$. To prove that I_1 has a limit we use formula (99) to find

$$\begin{aligned}
&\left\| t^{-1}\left(e^{Lt} c_0 - c_0 + \epsilon \int_0^t e^{L(t-\zeta)} v(\zeta)\, d\zeta \right) \right\|_{L^2} \\
&\quad \leqslant \left\| t^{-1}\big(e^{Lt} c_0 - c_0\big) \right\|_{L^2} + \epsilon e^{\omega t} \max_{0 \leqslant \zeta \leqslant t} \left\| v(\zeta) \right\|_{L^2}
\end{aligned}$$

(and because of (77)). The first term on the right has a limit as $t \to 0^+$ because e^{Lt} is a C_0-semigroup in L^2 and $c_0 \in H^3 \subset H^2 = \mathcal{D}(L)$ (see Lemma 5.1). The second term is clearly bounded when $t \to 0^+$. To verify the limit for the L^2 norm of the derivative we recall formula (94), yielding

$$\begin{aligned}
\left\| t^{-1}\big(c_x(t) - c_0\big) \right\|_{L^2} &\leqslant \left\| t^{-1}(ik)\big(e^{-(Dk^2 + ik\theta)t} - 1\big)\hat{c}_0 \right\|_{L^2} + \mu t \epsilon \max_{0 \leqslant \zeta \leqslant t} \left\| (ik)\hat{v}(k, \zeta) \right\|_{L^2} \\
&\quad + M \max_{0 \leqslant \zeta \leqslant t} \left\| (ik)\hat{c}(k, \zeta) \right\|_{L^2} \\
&\leqslant \left\| t^{-1}(ik)\big(e^{-(Dk^2 + ik\theta)t} - 1\big)\hat{c}_0 \right\|_{L^2} + \mu \epsilon \max_{0 \leqslant \zeta \leqslant 1} \left\| v(\zeta) \right\|_{H^1} \\
&\quad + M \max_{0 \leqslant \zeta \leqslant 1} \left\| c(\zeta) \right\|_{H^1},
\end{aligned}$$

as $t \in [0, 1]$. The last two terms are clearly bounded for $t \to 0^+$. To verify that the first term on the right also has a limit, we proceed as follows. Given that $|t^{-1}(1 - e^{-(Dk^2 + ik\theta)t})| \to |Dk^2 + i\theta k|$ as $t \to 0^+$ for all $k \in \mathbb{R}$, and since

$$k^2 \left|\hat{c}_0(k)\right|^2, k^4 \left|\hat{c}_0(k)\right|^2, k^6 \left|\hat{c}_0(k)\right|^2 \in L^1, \quad \text{because } c_0 \in H^3,$$

it follows from the dominated convergence theorem that

$$\begin{aligned}
\left\| t^{-1}(ik)\big(e^{-(Dk^2 + ik\theta)t} - 1\big)\hat{c}_0(\cdot) \right\|_{L^2}^2 &= \int_{\mathbb{R}} \left| t^{-1}\big(e^{-(Dk^2 + ik\theta)t} - 1\big) \right|^2 k^2 \left| c_0(k) \right|^2 dk \\
&\to \int_{\mathbb{R}} \left| Dk^2 + ik\theta \right|^2 k^2 \left| \hat{c}_0(k) \right|^2 < +\infty,
\end{aligned}$$

as $t \to 0^+$. Therefore $I_1(t)$ has a limit.

Likewise, we use representation formula (93) to estimate

$$\begin{aligned}
I_2(t) = \left\| t^{-1}\big(\hat{v}(t) - \hat{v}_0\big) \right\|_{H^1} &\leqslant \left\| t^{-1}\big(e^{-a(k)t} - 1\big)\hat{v}_0 \right\|_{H^1} + \left\| t^{-1} \int_0^t e^{-a(k)(t-\zeta)} \big(\tau'(\bar{c})c\big)\widehat{\ }(\cdot, \zeta)\, d\zeta \right\|_{H^1} \\
&\leqslant \left\| t^{-1}\big(e^{-a(k)t} - 1\big)\hat{v}_0 \right\|_{H^1} + M \max_{0 \leqslant \zeta \leqslant t} \left\| c(\zeta) \right\|_{H^1}.
\end{aligned}$$

The second term on the right is clearly bounded when $t \to 0^+$. To verify that the first term on the right has a limit, notice that $t^{-1}|1 - e^{-a(k)t}| \to |a(k)|$ as $t \to 0^+$ for all $k \neq 0$, and $|\hat{v}_0(k)|^2 \in L^1$, it follows from the dominated convergence theorem that

$$\left\|t^{-1}\left(e^{-a(\cdot)t} - 1\right)\hat{v}_0(\cdot)\right\|_{L^2}^2 = \int_{\mathbb{R}} \left|t^{-1}\left(e^{-a(k)t} - 1\right)\right|^2 \left|v_0(k)\right|^2 dk \to \int_{\mathbb{R}} \left|a(k)\right|^2 \left|\hat{v}_0(k)\right|^2,$$

as $t \to 0^+$, provided that the last integral exists. But this is exactly the case when $v_0 \in H^2 \cap L^1 \cap \mathcal{U}$. Indeed, since

$$\left|a(k)\right|^2 = \mu^{-2}\left(1 + s/k^2\right)^2 + \theta^2 k^2,$$

then it is clear from Corollary 5.11 and from $v_0 \in H^2 \subset H^1$ that

$$\int_{\mathbb{R}} \left|a(k)\right|^2 \left|\hat{v}_0(k)\right|^2 < +\infty,$$

and consequently $\|t^{-1}(e^{-a(\cdot)t} - 1)\hat{v}_0(\cdot)\|_{L^2}$ has a limit when $t \to 0^+$. In the same fashion we observe that

$$\left\|t^{-1}(ik)\left(e^{-a(k)t} - 1\right)\hat{v}_0\right\|_{L^2}^2 \to \int_{\mathbb{R}} \left|a(k)\right|^2 k^2 \left|\hat{v}_0(k)\right|^2 dk < +\infty,$$

as $t \to 0^+$, and in view that $\hat{v}_0 \in H^2 \cap L^1 \cap \mathcal{U}$.

Therefore $\|t^{-1}(e^{-a(k)t} - 1)\hat{v}_0\|_{H^1}$ and consequently $I_2(t)$ have a limit when $t \to 0^+$.

This shows that

$$\lim_{t \to 0^+} t^{-1}\left(\mathcal{S}(t)(v_0, c_0) - (v_0, c_0)\right)$$

exists in $H^1 \times H^1$, and the conclusion follows. $\square$

Remark 5.22. From the proof of last lemma, we verify that the set $H^3 \cap \mathcal{U} \cap L^1$ is contained in the domain of the infinitesimal generator of the C_0-semigroup $\{\tilde{\mathcal{S}}(t)\}_{t \geqslant 0}$ in H^1, constructed in Proposition 5.3, without previous knowledge of the generator itself (see Remark 5.4).

In view of Corollary 5.9 and Lemma 5.21, we notice that the set

$$\hat{\mathcal{D}} := \left(H^2 \cap L^1 \cap \mathcal{U}\right) \times H^3 \tag{108}$$

is dense in $H^1 \times H^1$; henceforth, and without loss of generality, we assume that $\mathcal{A}$ has domain $\hat{\mathcal{D}}$. Moreover, for each $(v, c) \in \hat{\mathcal{D}}$, $\mathcal{S}(t)(v, c) \in C^1((0, +\infty); H^1 \times H^1)$ and it satisfies (105).

Whence $\mathcal{A} : \hat{\mathcal{D}} \to H^1 \times H^1$ is closed and densely defined on the Hilbert space $H^1 \times H^1$, and we define its resolvent and spectra as

$$\rho(\mathcal{A}) := \left\{\lambda \in \mathbb{C}\colon\ \mathcal{A} - \lambda \text{ is one-to-one and onto, and } (\mathcal{A} - \lambda)^{-1} \text{ is bounded}\right\},$$

$$\sigma_{\mathrm{pt}}(\mathcal{A}) := \{\lambda \in \mathbb{C}\colon\ \mathcal{A} - \lambda \text{ is Fredholm with index 0 and has a non-trivial kernel}\},$$

$$\sigma_{\mathrm{ess}}(\mathcal{A}) := \mathbb{C} \backslash \{\lambda \in \mathbb{C}\colon\ \mathcal{A} - \lambda \text{ is Fredholm with index 0}\}.$$

In accordance to these definitions we also have that

$$\sigma(\mathcal{A}) := \sigma_{\mathrm{pt}}(\mathcal{A}) \cup \sigma_{\mathrm{ess}}(\mathcal{A}) = \mathbb{C}\backslash\rho$$

(see Kato [23, p. 167]). We recall the Fredholm parameters of $\mathcal{A}$, namely,

$$\begin{aligned} \mathrm{nul}(\mathcal{A}) &= \dim\ker(\mathcal{A}),\\ \mathrm{def}(\mathcal{A}) &= \mathrm{codim}\,\mathcal{R}(\mathcal{A}),\\ \mathrm{ind}(\mathcal{A}) &= \mathrm{nul}(\mathcal{A}) - \mathrm{def}(\mathcal{A}). \end{aligned}$$

Remark 5.23. If we take $w, v \in H^2 \cap L^1 \cap \mathcal{U}$, so that $W = \mathcal{J}w \in H^2$ with $W_{xx} = w \in H^4$, then integrating by parts we obtain

$$\langle w, \mathcal{J}v\rangle_{L^2} = \int_{\mathbb{R}} W_{xx}^* \mathcal{J}v\,dx = \int_{\mathbb{R}} W^*(\mathcal{J}v)_{xx}\,dx = \int_{\mathbb{R}} (\mathcal{J}w)^* v\,dx = \langle \mathcal{J}w, v\rangle_{L^2},$$

and whence, $\langle w_x, \mathcal{J}v_x\rangle_{L^2} = \langle \mathcal{J}w_x, v_x\rangle_{L^2}$ as well. That is, $\mathcal{J}$ is self-adjoint, $\mathcal{J}^* = \mathcal{J}$ as an operator $H^1 \to H^1$.

Thanks to last remark we readily note that the Hilbert space adjoint $\mathcal{A}^* : \hat{\mathcal{D}} \to H^1 \times H^1$ can be explicitly written as

$$\big(\mathcal{A}^*(v,c)\big)^\top = \begin{pmatrix} (s/\mu)\mathcal{J} - 1/\mu + \theta\partial_x & \epsilon \\ \tau'(\bar{c})/\mu & D\partial_x^2 + \theta\partial_x + R'(\bar{c}) \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix}. \tag{109}$$

Notice that $\mathcal{A}^*$ is also densely defined with domain $\mathcal{D}(\mathcal{A}^*) = \hat{\mathcal{D}}$.

The connection between the spectrum σ of problem (14) and $\sigma(\mathcal{A})$ is now provided by the following lemma.

Lemma 5.24. *For all $\lambda \in \mathbb{C}$ and fixed $\epsilon \geqslant 0$,*

$$\mathrm{nul}(\mathcal{A} - \lambda) \leqslant \mathrm{nul}\big(\mathcal{T}^\epsilon(\lambda)\big), \tag{110}$$

$$\mathrm{nul}\big(\mathcal{A}^* - \lambda^*\big) \leqslant \mathrm{nul}\big(\mathcal{T}^\epsilon(\lambda)^*\big). \tag{111}$$

Proof. To show (110), suppose $(v,c) \in \ker(\mathcal{A} - \lambda) \subset \hat{\mathcal{D}}$. This implies that

$$\frac{d}{dt}\big(\mathcal{S}(t)(v,c)\big) = \mathcal{S}(t)\mathcal{A}(v,c) = \lambda\mathcal{S}(t)(v,c),$$

or, in other words, that $\mathcal{S}(t)(v,c) = e^{\lambda t}(v,c)$. Hence, $e^{\lambda t}(u,c)$ solves the system (92), which is equivalent to say that (v,c) solves system (107). Since $(v,c) \in \hat{\mathcal{D}}$, one may define $u = \int_{-\infty}^x v$ so that $u_x = v$, and $(u,c) \in H^3 \times H^3$ satisfies (14). If we denote

$$W := (u, u_x, u_{xx}, c, c_x)^\top \in H^1\big(\mathbb{R}; \mathbb{C}^5\big),$$

then it solves $\mathcal{T}^\epsilon(\lambda)W = 0$. This implies that $W \in \ker\mathcal{T}^\epsilon(\lambda)$. Consider the linear mapping

$$\Phi : \ker(\mathcal{A}-\lambda) \to \ker \mathcal{T}^{\epsilon}(\lambda),$$
$$\Phi(v,c) := W.$$

Φ is clearly an injective mapping because $\Phi(v,c)=0$ implies $(v,c)=0$. Therefore (110) holds, as claimed.

To show inequality (111), assume $(v,c) \in \ker(\mathcal{A}^*-\lambda^*) \subset \hat{\mathcal{D}}$. Then, from the expression (109), we arrive at the system

$$\begin{aligned}(s/\mu)\mathcal{J}v - v/\mu + \theta v_x + \epsilon c &= \lambda^* v,\\ \big(\tau'(\bar{c})/\mu\big)v + Dc_{xx} + \theta c_x + R'(\bar{c})c &= \lambda^* c.\end{aligned} \tag{112}$$

Since $(v,c) \in \hat{\mathcal{D}} = (H^2 \cap \mathcal{U}) \times H^3$, we can define $\eta := \mathcal{J}v \in H^4$, with $\eta_{xx} = v$. Writing last system in terms of η we arrive at

$$\begin{aligned}s\eta - \big(1+\mu\lambda^*\big)\eta_{xx} + \mu\theta\eta_{xxx} + \mu\epsilon c &= 0,\\ Dc_{xx} + \theta c_x + \big(R'(\bar{c}) - \lambda^*\big)c + \big(\tau'(\bar{c})/\mu\big)\eta_{xx} &= 0.\end{aligned} \tag{113}$$

We readily recognize that $(\tilde{u},\tilde{e}) := (\eta, -\mu Dc/s) \in H^4 \times H^2$, is a solution to the system of equations (36). By Remark 4.1, $Y = (\tilde{u},\tilde{v},\tilde{w},\tilde{c},\tilde{e})^\top \in H^1(\mathbb{R};\mathbb{C}^5)$ (where $\tilde{v}$, $\tilde{w}$ and $\tilde{c}$ are defined in (37)) belongs to $\ker \mathcal{T}^{\epsilon}(\lambda)^*$. The linear mapping

$$\Psi : \ker\big(\mathcal{A}^*-\lambda^*\big) \to \ker \mathcal{T}^{\epsilon}(\lambda)^*,$$
$$\Psi(v,c) := Y$$

is clearly injective because $\Psi(v,c)=0$ implies $(v,c)=0$. Therefore (111) holds and the lemma is proved. □

Corollary 5.25. *For all $\epsilon \geqslant 0$,*

$$\sigma_{\mathrm{pt}}(\mathcal{A}) \subseteq \sigma, \tag{114}$$

$$\sigma_{\mathrm{ess}}(\mathcal{A}) \subseteq \sigma. \tag{115}$$

Proof. Suppose $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{A})$. Then, $\mathcal{A}-\lambda$ has a non-trivial kernel and (110) implies that $\mathcal{T}^{\epsilon}(\lambda)$ has a non-trivial kernel in $H^1(\mathbb{R};\mathbb{C}^5)$; therefore, $\lambda \in \mathbb{C}\backslash\rho = \sigma$. This shows (114).

To prove (115), suppose that $\lambda \in \rho = \mathbb{C}\backslash\sigma$. This implies that $\mathcal{T}^{\epsilon}(\lambda)$ is Fredholm with index 0, with $\operatorname{nul}\mathcal{T}^{\epsilon}(\lambda) = \operatorname{def}\mathcal{T}^{\epsilon}(\lambda) = 0$ (if the deficiency is finite, then the range is closed). Moreover, $\mathcal{R}(\mathcal{T}^{\epsilon}(\lambda))^{\perp} = \ker \mathcal{T}^{\epsilon}(\lambda)^*$. Therefore by inequality (111), $0 = \operatorname{def}\mathcal{T}^{\epsilon}(\lambda) = \operatorname{nul}(\mathcal{T}^{\epsilon}(\lambda)^*) \geqslant \operatorname{nul}(\mathcal{A}^* - \lambda^*) = \operatorname{def}(\mathcal{A}-\lambda)$. This shows that $\operatorname{def}(\mathcal{A}-\lambda)=0$ and $\mathcal{R}(\mathcal{A}-\lambda)$ is closed. In addition, inequality (110) implies that $\operatorname{nul}(\mathcal{A}-\lambda)=0$. Therefore $\mathcal{A}-\lambda$ is Fredholm with index 0, that is, $\lambda \in \mathbb{C}\backslash\sigma_{\mathrm{ess}}(\mathcal{A})$. □

Corollary 5.26. *For $\epsilon \geqslant 0$ sufficiently small,*

$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda < 0\} \cup \{0\}, \tag{116}$$

that is, $\mathcal{A}$ is spectrally stable, being $\lambda = 0$ an isolated simple eigenvalue with associated eigenfunction $(\bar{u}_{xx}, \bar{c}_x) \in \hat{\mathcal{D}} \subset H^1 \times H^1$.

Proof. The first assertion is an immediate consequence of Corollary 5.25 and Theorem 1. That $(\bar{u}_{xx}, \bar{c}_x)$ is the eigenfunction of $\mathcal{A}$ associated to $\lambda = 0$ follows from the equations for the traveling waves (7). Differentiate the second equation in (7) to verify that $(\bar{u}_{xx}, \bar{c}_x)$ satisfies (107) with $\lambda = 0$. We readily note that its geometric multiplicity is $g.m. = 1$ because of inequality (110) and Theorem 1. That the algebraic multiplicities are the same follows from the next simple observation. Suppose, by contradiction, that there exists $0 \neq (v, c) \in \hat{\mathcal{D}}$ such that $\mathcal{A}(v, c) = (\bar{u}_{xx}, \bar{c}_x)$. Since v has double zero-mass we can define $u := \int_{-\infty}^{x} v$ and set

$$W := (u, u_x, u_{xx}, c, c_x)^\top \in H^1\big(\mathbb{R}; \mathbb{C}^5\big).$$

But the fact that $\mathcal{A}(v, c) = (\bar{u}_{xx}, \bar{c}_x)$ leads to the system of equations

$$s\mathcal{J}v - v - \mu\theta v_x + \tau'(\bar{c})c = \bar{u}_{xx},$$
$$\epsilon v + Dc_{xx} - \theta c_x + R'(\bar{c})c = \bar{c}_x.$$

Differentiating first equation yields

$$su - u_{xx} - \mu\theta u_{xxx} + \big(\tau'(\bar{c})c\big)_x = \bar{u}_{xxx},$$
$$\epsilon u_x + Dc_{xx} - \theta c_x + R'(\bar{c})c = \bar{c}_x,$$

which is equivalent to

$$\mathcal{T}^\epsilon(0)W = \tilde{\mathbb{A}}_1^\epsilon(x)\bar{W},$$

with $\bar{W} = (\bar{u}_x, \bar{u}_{xx}, \bar{u}_{xxx}, \bar{c}_x, \bar{c}_{xx})^\top$. This is a contradiction with the fact that $\lambda = 0 \in \sigma_{\text{pt}}$ has algebraic multiplicity equal to one as an eigenvalue of problem (14) (see Definition 2.7). This shows that $\lambda = 0$ is a simple isolated eigenvalue of $\mathcal{A}$. $\square$

Remark 5.27. It is well known that if $\lambda \in \mathbb{C}$ is an isolated eigenvalue of $\mathcal{A}$ then λ^* is an eigenvalue of $\mathcal{A}^*$ with the same geometric and algebraic multiplicities [23]. Therefore there exists an adjoint eigenfunction $(\bar{\psi}, \bar{\phi}) \in \hat{\mathcal{D}}$ such that

$$\mathcal{A}^*(\bar{\psi}, \bar{\phi}) = 0,$$

and $\lambda = 0$ is a simple isolated eigenvalue of $\mathcal{A}^*$ by the previous results.

Remark 5.28. Since on a reflexive Banach space, weak and weak* topologies coincide, the dual semigroup $\{\mathcal{S}(t)^*\}_{t\geqslant 0}$ consisting of all adjoint operators on $H^1 \times H^1$ is a C_0-semigroup (see [9, p. 44]). Moreover, the infinitesimal generator the dual semigroup is $\mathcal{A}^*$ (see Corollary 10.6 in [32]). By properties of C_0-semigroups and their generators we have

$$\mathcal{S}(t)(\bar{u}_{xx}, \bar{c}_x) = (\bar{u}_{xx}, \bar{c}_x),$$
$$\mathcal{S}(t)^*(\bar{\psi}, \bar{\phi}) = (\bar{\psi}, \bar{\phi}), \tag{117}$$

for all $t \geqslant 0$.

Denote the inner product[3]

$$\Theta := \big\langle (\bar{u}_{xx}, \bar{c}_x), (\bar{\psi}, \bar{\phi}) \big\rangle_{H^1 \times H^1} \neq 0.$$

We then introduce the $\mathcal{A}$-invariant subspace

$$X_1 \subset H^1 \times H^1,$$

defined as the range of the spectral projection

$$\mathcal{P}_1(v, c) := (v, c) - \frac{\langle (v, c), (\bar{\psi}, \bar{\phi}) \rangle_{H^1 \times H^1}}{\Theta} (\bar{u}_{xx}, \bar{c}_x), \tag{118}$$

for all $(v, c) \in H^1 \times H^1$. From its definition and in view of (117), we readily observe that $\mathcal{P}_1$ commutes with the semigroup,

$$\mathcal{P}_1 \mathcal{S}(t) = \mathcal{S}(t) \mathcal{P}_1,$$

for all $t \geqslant 0$. This implies that X_1 is an $\{\mathcal{S}(t)\}_{t \geqslant 0}$-invariant closed subspace of $H^1 \times H^1$. We introduce the operator $\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subset X_1 \to X_1$ as the restriction $\mathcal{A}_1 = \mathcal{A}_{|X_1}$, or more precisely,

$$\begin{aligned} &\mathcal{A}_1(v, c) := \mathcal{A}(v, c), \\ &\quad \text{for all } (v, c) \in \mathcal{D}(\mathcal{A}_1) := \big\{ (v, c) \in \hat{\mathcal{D}} \cap X_1 \colon \mathcal{A}(v, c) \in X_1 \big\}. \end{aligned} \tag{119}$$

In this fashion, we project out the eigenspace spanned by $(\bar{u}_{xx}, \bar{c}_x)$ and rule out $\lambda = 0$ from the spectrum. Whence, we obtain the following immediate

Corollary 5.29. *$\sigma(\mathcal{A}_1)$ is a strict subset of the left-half complex plane,*

$$\sigma(\mathcal{A}_1) \subset \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda < 0\},$$

and, consequently, the spectral bound of $\mathcal{A}_1$ is negative, $s(\mathcal{A}_1) < 0$.

Lemma 5.30. *The family of operators $\{\mathcal{S}_1(t)\}_{t \geqslant 0}$, defined as*

$$\begin{aligned} &\mathcal{S}_1(t) : X_1 \to X_1, \\ &\mathcal{S}_1(t)(v, c) := \mathcal{S}(t) \mathcal{P}_1(v, c), \end{aligned} \tag{120}$$

for all $(v, c) \in X_1$, constitutes a C_0-semigroup in X_1 with infinitesimal generator $\mathcal{A}_1$.

Proof. The semigroup properties (68)–(71) in X_1 are inherited by those of $\mathcal{S}(t)$ in $H^1 \times H^1$. Since X_1 is an $\{\mathcal{S}(t)\}_{t \geqslant 0}$-invariant closed subspace of $H^1 \times H^1$, the second assertion follows from the corollary in Section 2.2 of [9, pp. 61–62]. $\square$

[3] The product is non-vanishing because otherwise $(\bar{\psi}, \bar{\phi}) \in (\ker \mathcal{A})^{\perp} = \mathcal{R}(\mathcal{A}^*)$ and therefore $(\mathcal{A}^*)^2(v, c) = 0$ for some $0 \neq (v, c) \in \hat{\mathcal{D}}$, a contradiction with $\lambda = 0$ being a simple isolated eigenvalue of $\mathcal{A}^*$.

5.4. Resolvent bounds and semigroup decay estimates

In view of the previous results, Gearhart–Prüss theorem [18,35,9] on the Hilbert space X_1 provides uniform asymptotic stability with exponential decay of the semigroup $\mathcal{S}_1(t)$ provided we are able to prove that the resolvent of $\mathcal{A}_1$ is uniformly bounded in $\operatorname{Re}\lambda > 0$ (condition (75) holds). This task is already substantially completed as $\sigma(\mathcal{A}_1)$ is localized in the strict left-half complex plane, so that $\|(\mathcal{A}_1 - \lambda)^{-1}\|$ (in the operator $X_1 \to X_1$ norm) is uniformly bounded for each $|\lambda| \leqslant \delta$, $\operatorname{Re}\lambda \geqslant 0$, with $\delta > 0$ finite. Notice that $\lambda = 0$ is already excluded from the spectrum of $\mathcal{A}_1$. Therefore, it suffices to verify the resolvent bounds for $\operatorname{Re}\lambda > 0$, with $|\lambda| \gg 1$ sufficiently large.

For that purpose we establish the following resolvent bounds whose proof can be found in Appendix B.

Proposition 5.31. (i) *For each $\epsilon \geqslant 0$ there exist uniform positive constants C_1, $K_1 > 0$, such that*

$$\left\|(\lambda - \mathcal{A})^{-1}\right\|_{H^1 \to H^1} \leqslant C_1, \tag{121}$$

for $\operatorname{Re}\lambda \geqslant 0$ *and* $|\operatorname{Im}\lambda| \geqslant K_1 > 0$.

(ii) *There exists $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$ we can find uniform positive constants C_2, $K_2 > 0$ such that*

$$\left\|(\lambda - \mathcal{A})^{-1}\right\|_{H^1 \to H^1} \leqslant \frac{C_2}{\operatorname{Re}\lambda}, \tag{122}$$

for all $\operatorname{Re}\lambda \geqslant K_2 > 0$.

Proof. See Appendix B. $\square$

Remark 5.32. Observe that bounds (121) and (122) are independent of each other, that is, (122) holds for all $\operatorname{Re}\lambda \gg 1$ sufficiently large provided that ϵ is sufficiently small, independently of $\operatorname{Im}\lambda$, whereas (121) holds for all $\operatorname{Re}\lambda \geqslant 0$, $\epsilon \geqslant 0$ and $|\operatorname{Im}\lambda|$ large enough. Likewise, observe that uniform boundedness of the point spectrum in $\{\operatorname{Re}\lambda \geqslant -\frac{1}{2\mu}\}$ (see Lemma 4.4) precludes the existence of a sequence λ_n in σ_{pt}, with $\operatorname{Re}\lambda_n < 0$ such that $|\operatorname{Im}\lambda_n| \to +\infty$ and $\operatorname{Re}\lambda_n \to 0^-$, in accordance with the uniform bound (121) for $|\operatorname{Im}\lambda|$ large enough.

Lemma 5.33 *(Semigroup decay rates). For each $\epsilon \geqslant 0$ sufficiently small, there exist constants $M_0 \geqslant 1$ and $\omega_0 > 0$ such that the C_0-semigroup $\{\mathcal{S}_1(t)\}_{t\geqslant 0}$ on X_1 satisfies*

$$\left\|\mathcal{S}_1(t)(v, c)\right\|_{H^1 \times H^1} \leqslant M_0 e^{-\omega_0 t}\left\|(v, c)\right\|_{H^1 \times H^1}, \tag{123}$$

for all $t \geqslant 0$, $(v, c) \in X_1$.

Proof. Let $\epsilon \geqslant 0$ be sufficiently small so that the conclusion of Proposition 5.31(ii) holds. Let $R_1 \geqslant \max\{K_1, K_2\} > 0$, where $K_i > 0$ are as in Proposition 5.31 above. Therefore estimate (122) holds for all $\operatorname{Re}\lambda \geqslant R_1$ and any value of $\operatorname{Im}\lambda$, and estimate (121) holds for $|\operatorname{Im}\lambda| \geqslant R_1$ and any value of $\operatorname{Re}\lambda \geqslant 0$. This implies that we can find a radius $R_2 > R_1 > 0$ sufficiently large such that there holds the uniform estimate

$$\left\|(\lambda - \mathcal{A})^{-1}\right\|_{H^1 \to H^1} \leqslant C,$$

for all $|\lambda| \geqslant R_2$, $\operatorname{Re}\lambda \geqslant 0$. Since $\mathcal{A}_1 = \mathcal{A}$ on $X_1 \subset H^1 \times H^1$, this implies that $\|(\lambda - \mathcal{A}_1)^{-1}\|$ is uniformly bounded outside the half circle $|\lambda| \leqslant R_2$, $\operatorname{Re}\lambda \geqslant 0$. By spectral stability (Corollary 5.29) we know that the resolvent of $\mathcal{A}_1$ is uniformly bounded inside the intersection of any ball of finite radius and $\operatorname{Re}\lambda \geqslant 0$. Whence we conclude that

$$\sup_{\operatorname{Re}\lambda>0} \left\|(\lambda - \mathcal{A}_1)^{-1}\right\|_{X_1 \to X_1} < +\infty. \tag{124}$$

In view of Corollary 5.29 and the uniform bound (124), a direct application of Gearhart–Prüss theorem [18,35,9] to operator $\mathcal{A}_1$ on the Hilbert space X_1 implies that the semigroup $\{S_1(t)\}_{t\geqslant 0}$ is uniformly exponentially stable, and that (123) holds for some $M_0 \geqslant 1$ and some $\omega_0 > 0$. $\square$

We are now ready to prove the linear stability result.

5.5. *Proof of Theorem 2*

Assume $\epsilon \geqslant 0$ is sufficiently small. If $(u_0, c_0) \in H^2 \times H^1$ then by Lemma 5.14 we can solve (22) with initial condition (24), such that $(v, c)(t) \in C([0, +\infty); H^1 \times H^1)$ is the global mild solution determined uniquely by (93) and (94). Projecting onto the closed subspace X_1 we have by Lemma 5.33 that

$$\left\|\mathcal{P}_1(v, c)(t)\right\|_{H^1\times H^1} = \left\|(v, c)(t) - \alpha_*(\bar{u}_{xx}, \bar{c}_x)\right\|_{H^1\times H^1} \leqslant Ce^{-\omega_0 t}, \tag{125}$$

where $\alpha_* \in \mathbb{R}$ is given by

$$\alpha_* = \Theta^{-1}\big\langle (v_0, c_0), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1}. \tag{126}$$

This shows the first assertion of Theorem 2.

To prove the second part of the theorem, assume $(u_0, c_0) \in (W^{1,1} \cap H^3) \times H^2$. Then by Lemma 5.18, the initial condition $(v_0, c_0) = (u_{0x}, c_0) \in (L^1 \cap H^2) \times H^2$ determines uniquely a global strong solution $(v, c) \in C([0, +\infty); H^1 \times H^1)$, with $(v, c)(t) \in H^2 \times H^2$ for each $t > 0$. Moreover, by Proposition 5.20, $v(t)$ has zero-mass for each $t > 0$ and we can define its antiderivative u by (101), with $u_x = v$ a.e. in x. Since $v(t) \in H^2$, then $u(t) \in H^3$; therefore, $(u, c) \in C([0, +\infty); H^1 \times H^1)$, with $(u, c)(t) \in H^3 \times H^2$ for each $t > 0$, and we have a strong solution of the original linear system (13) with initial condition $(u, c)(0) = (u_0, c_0)$.

Finally, to get (27) we notice that

$$\left\|v(t)\right\|_{L^2} \leqslant \left\|v(t) - \alpha_*\bar{u}_{xx}\right\|_{L^2} + \|\alpha_*\bar{u}_{xx}\|_{L^2} \leqslant \tilde{C}\big(e^{-\omega_0 t} + 1\big) \leqslant C_1,$$

for all $t \geqslant 1$ large. Thus, $\|v(t)\|_{L^2}$ is bounded independently of t for t large. Since $\hat{u}(k, t) = -(i/k)\hat{v}(k, t)$, Lemmas 5.17(c) and 5.5(c) imply that $\|u(t)\|_{L^2} \leqslant C_2$ is bounded independently of t, as long as $t \geqslant 1$ is large. These estimates, together with (125), yield

$$\left\|c(t) - \alpha_*\bar{c}_x\right\|_{L^\infty} \leqslant \left\|c(t) - \alpha_*\bar{c}_x\right\|_{L^2}\left\|c_x(t) - \alpha_*\bar{c}_{xx}\right\|_{L^2} \leqslant Ce^{-\omega_0 t} \to 0,$$

and

$$\begin{aligned}
\left\|u(t) - \alpha_*\bar{u}_x\right\|_{L^\infty} &\leqslant \left\|u(t) - \alpha_*\bar{u}_x\right\|_{L^2}\left\|u_x(t) - \alpha_*\bar{u}_{xx}\right\|_{L^2} \\
&= \left\|u(t) - \alpha_*\bar{u}_x\right\|_{L^2}\left\|v(t) - \alpha_*\bar{u}_{xx}\right\|_{L^2} \\
&\leqslant Ce^{-\omega_0 t}\big(\left\|u(t)\right\|_{L^2} + \tilde{C}\|\bar{c}_x\|_{L^2}\big) \\
&\leqslant CC_2e^{-\omega_0 t} \to 0,
\end{aligned}$$

as $t \to +\infty$. We conclude the proof of Theorem 2.

6. Nonlinear orbital asymptotic stability

In this section we study the nonlinear asymptotic behavior of the traveling waves, which leads to the proof of Theorem 3. Following Pego and Weinstein [33], we seek to represent the solution to (1) in the form of a dominant phase-modulated traveling wave plus a perturbation. We perturb the traveling waves by imposing initial conditions sufficiently close to a translated wave, hoping that this would yield a slight change in phase. Unlike the case of solitary waves for equations of KdV-type [33], in the present context the wave speed is uniquely determined, and therefore our family of translates is one-dimensional, making the analysis considerably simpler. (In other words, there is no Jordan block associated to the eigenvalue $\lambda = 0$.)

As before, we assume that $(\bar{u}, \bar{c})(x + \theta t)$ is the wave pair solution to (1) for a certain sufficiently small parameter value $\epsilon > 0$, for which spectral and linear stability hold. Once again, we drop the superscript notation and the dependence of ϵ of the waves for convenience.

6.1. The ansatz

We look for representing a solution to the original nonlinear system (1), of the form

$$(\tilde{u}, \tilde{c})(x, t) = (u, c)\big(x + \theta t + \alpha(t), t\big) + (\bar{u}, \bar{c})\big(x + \theta t + \alpha(t)\big), \tag{9}$$

with initial condition

$$(\tilde{u}, \tilde{c})(x, 0) = (u, c)(x, 0) + (\bar{u}, \bar{c})(x + \alpha_0), \tag{127}$$

where $\alpha_0 \in \mathbb{R}$ is fixed, representing a translate of the traveling waves. We expect the wave to adjust to a nearby traveling wave for which the time-varying "shift" or phase parameter $\alpha(t)$ is allowed to modulate. In this fashion, we expect the perturbation (u, c) to decay in time to provide stability. In view that the decaying constructed semigroup $\mathcal{S}_1(t)$ is defined in terms of the deformation gradient $v = u_x$, we start with a formulation in the $(v, c) = (u_x, c)$ variables and look for a decomposition of the form

$$(\tilde{v}, \tilde{c})(x, t) = (v, c)\big(x + \theta t + \alpha(t), t\big) + (\bar{u}_x, \bar{c})\big(x + \theta t + \alpha(t)\big). \tag{128}$$

In order to achieve exponential decay of the perturbation (v, c) we need to construct the phase $\alpha(t)$ in such a way that $(v, c)(\cdot, t) \in X_1 = \mathcal{R}(\mathcal{P}_1)$, which will be satisfied by modulating the parameter $\alpha(t)$ in a time-depending fashion. This requirement will lead to the modulation evolution equation for α. Hence, we construct the phase such that

$$\mathcal{G}\big[(\tilde{v}, \tilde{c}), \alpha\big](t) := \big\langle (\tilde{v}, \tilde{c})(\cdot, t) - (\bar{u}_x, \bar{c})\big(\cdot + \theta t + \alpha(t)\big), (\bar{\psi}, \bar{\phi})\big(\cdot + \theta t + \alpha(t)\big)\big\rangle_{H^1 \times H^1} = 0, \tag{129}$$

for all t. This ensures that $(v, c)(\cdot, t) \in X_1$.

We start by proving the local existence of such a decomposition.

Lemma 6.1. *Let $T_1 \geqslant 0$. Then there exist $\delta_0, \delta_1 > 0$ such that for any $\alpha_0 \in \mathbb{R}$, if $(\tilde{v}, \tilde{c}) \in C([0, T_1]; H^1 \times H^1)$ is such that*

$$\sup_{0 \leqslant t \leqslant T_1} \big\| (\tilde{v}, \tilde{c})(\cdot, t) - (\bar{u}_x, \bar{c})(\cdot + \theta t + \alpha_0) \big\|_{H^1 \times H^1} < \delta_0, \tag{130}$$

then there exists a function $t \mapsto \alpha(t)$, $\alpha \in C([0, T_1])$, with

$$\sup_{0 \leqslant t \leqslant T_1} \big| \alpha(t) - \alpha_0 \big| < \delta_1, \tag{131}$$

such that

$$\mathcal{G}\big[(\tilde{v},\tilde{c}),\alpha\big](t)=0, \tag{132}$$

for all $t \in [0,T_1]$. Moreover, α depends analytically on $(\tilde{v},\tilde{c})$ and if $(\tilde{v},\tilde{c})$ is differentiable a.e. in $t \in [0,T_1]$, so is α.

Proof. We follow [33] closely, with considerable simplifications due to the one-dimensionality of the manifold of wave solutions. Without loss of generality we can assume $\alpha_0 = 0$. Since the parameter family of waves is generated by simple translation, which defines an analytic mapping $\alpha \mapsto (\bar{u}_x,\bar{c})(\cdot+\alpha)$, it is not hard to verify that the mapping $\mathcal{G}$ defined in (129) is analytic from a neighborhood of

$$\begin{gathered} V_0 \in C\big([0,T_1];H^1\times H^1\big)\times C\big([0,T_1]\big),\\ t\mapsto V_0(t),\quad V_0(t):=\big((\bar{u}_x,\bar{c})\big(\cdot+\theta t+\alpha(t)\big),0\big), \end{gathered}$$

to $C([0,T_1])$. Observe that, by the choice of V_0, then $\mathcal{G}[V_0]=0$. Writing

$$\begin{aligned} \mathcal{G}\big[(\tilde{v},\tilde{c}),\alpha\big](t) &= \big\langle(\tilde{v},\tilde{c})(\cdot,t)-(\bar{u}_x,\bar{c})\big(\cdot+\theta t+\alpha(t)\big),(\bar{\psi},\bar{\phi})\big(\cdot+\theta t+\alpha(t)\big)\big\rangle_{H^1\times H^1}\\ &= \int_{\mathbb{R}}\big\langle(\tilde{v},\tilde{c})(x,t),(\bar{\psi},\bar{\phi})\big(x+\theta t+\alpha(t)\big)\big\rangle\,dx\\ &\quad -\int_{\mathbb{R}}\big\langle(\bar{u}_x,\bar{c})\big(x+\theta t+\alpha(t)\big),(\bar{\psi},\bar{\phi})\big(x+\theta t+\alpha(t)\big)\big\rangle\,dx\\ &\quad +\int_{\mathbb{R}}\big\langle(\tilde{v}_x,\tilde{c}_x)(x,t),(\bar{\psi}_x,\bar{\phi}_x)\big(x+\theta t+\alpha(t)\big)\big\rangle\,dx\\ &\quad -\int_{\mathbb{R}}\big\langle(\bar{u}_{xx},\bar{c}_x)\big(x+\theta t+\alpha(t)\big),(\bar{\psi}_x,\bar{\phi}_x)\big(x+\theta t+\alpha(t)\big)\big\rangle\,dx \end{aligned}$$

(where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$), we compute the Fréchet derivative of $\mathcal{G}$ with respect to α at V_0 to find that

$$\begin{aligned} \frac{\partial\mathcal{G}}{\partial\alpha}[V_0](\delta\alpha)(t) &= -\int_{\mathbb{R}}\big\langle(\bar{u}_{xx},\bar{c}_x)(x+\theta t+\alpha_0),(\bar{\psi},\bar{\phi})(x+\theta t+\alpha_0)\big\rangle\,dx(\delta\alpha)(t)\\ &\quad -\int_{\mathbb{R}}\big\langle(\bar{u}_{xxx},\bar{c}_{xx})(x+\theta t+\alpha_0),(\bar{\psi}_x,\bar{\phi}_x)(x+\theta t+\alpha_0)\big\rangle\,dx(\delta\alpha)(t)\\ &= -\big\langle(\bar{u}_{xx},\bar{c}_x),(\bar{\psi},\bar{\phi})\big\rangle_{H^1\times H^1}(\delta\alpha)(t)\\ &= -\Theta(\delta\alpha)(t), \end{aligned}$$

for each $t \in [0,T_1]$. Therefore, since

$$\frac{\partial\mathcal{G}}{\partial\alpha}[V_0]=-\Theta\neq 0,$$

we apply the generalized implicit function theorem (see [29, Chapter 6]) to conclude the existence of $\alpha \in C([0, T_1])$ satisfying (131) for some $\delta_1 > 0$, and such that (132) holds for all $t \in [0, T_1]$. Moreover, the mapping $(\tilde{v}, \tilde{c}) \mapsto \alpha$ is analytic from the subset in $C([0, T_1]; H^1 \times H^1)$ such that (130) holds, to the subset of $C([0, T_1])$ such that (131) holds.

Finally, to verify that α inherits the same differentiability in t of $(\tilde{v}, \tilde{c})$, observe that for $0 < t_0 < T_1$, the curve $h \mapsto (\tilde{v}, \tilde{c})(t_0 + h)$ is differentiable a.e. with values in $C([0, T_1]; H^1 \times H^1)$ provided that $h \in [0, T_1 - t_0)$, and that $(\tilde{v}, \tilde{c})(t)$ is differentiable a.e. in $[0, T_1]$. Since the maps $(\tilde{v}, \tilde{c})(t) \mapsto \alpha(t)$ and $(\tilde{v}, \tilde{c})(t+h) \mapsto \alpha(t+h)$ are analytic for each $t < t_0$, $h \in (0, T_1 - t_0)$, we conclude that α is differentiable a.e. in $0 < t < T_1$. $\square$

We will now show that the decomposition can be extended to later times.

Lemma 6.2. *There exists $\delta_0 > 0$ such that, if, for some $T_0 > 0$,*

$$\sup_{0 \leqslant t \leqslant T_0} \left\| (\tilde{v}, \tilde{c})(\cdot, t) - (\bar{u}_x, \bar{c})\big(\cdot + \theta t + \alpha(t)\big) \right\|_{H^1 \times H^1} < \frac{1}{2}\delta_0, \tag{133}$$

and if $\alpha \in C([0, T_0])$, together with

$$\mathcal{G}\big[(\tilde{v}, \tilde{c}), \alpha\big](t) = 0, \tag{134}$$

for all $t \in [0, T_0]$, then there exists a unique extension $\alpha \in C([0, T_0 + T_*])$ for some $T_* > 0$, satisfying (134) *also in $t \in [0, T_0 + T_*]$ and with the same differentiability.*

Proof. Let $\delta_0 > 0$ be given by Lemma 6.1 for the same $T_1 > 0$. Put

$$(\check{v}, \check{c})(x, t) := (\tilde{v}, \tilde{c})(x, t + T_0),$$
$$\check{\alpha}_0 := \theta T_0 + \alpha(t_0).$$

Then, by (133),

$$\begin{aligned} \left\| (\check{v}, \check{c})(\cdot, 0) - (\bar{u}_x, \bar{c})(\cdot + \check{\alpha}_0) \right\|_{H^1 \times H^1} &= \left\| (\tilde{v}, \tilde{c})(\cdot, T_0) - (\bar{u}_x, \bar{c})\big(\cdot + \theta_0 + \alpha(T_0)\big) \right\|_{H^1 \times H^1} \\ &< \frac{1}{2}\delta_0 < \delta_0. \end{aligned}$$

Thus, by continuity, there exists some $T_* > 0$ sufficiently small such that $(\check{v}, \check{c})(\cdot, t)$ satisfies the hypotheses of Lemma 6.1 for $t \in [0, T_*]$. Choose $T_1 = T_*$, and therefore one obtains $\check{\alpha}(t)$ which satisfies $\check{\alpha}(0) = \check{\alpha}_0 = \theta T_0 + \alpha(T_0)$ and the constraint (134), for $t \in [0, T_*]$. The extension is thus defined as

$$\alpha(t) := \check{\alpha}(t - T_0) - \check{\alpha}_0 + \alpha(T_0),$$

for $t \in [T_0, T_0 + T_*]$. Differentiability follows from Lemma 6.1 as well. $\square$

6.2. The modulated equations

Let us now see how to derive the modulation and perturbation equations. Consider initial conditions $(\tilde{v}_0, \tilde{c}_0) \in H^2 \times H^2$ to the nonlinear system (23), written here in integrated form

$$\begin{aligned} \mu v_t + v - s\mathcal{J}v &= \tau(c) - \tau(0), \\ c_t - Dc_{xx} - \epsilon v &= R(c), \end{aligned} \tag{135}$$

where $\mathcal{J}$ is the densely defined operator in (91). By Proposition A.2, there exists a unique strong solution $(\tilde{v}, \tilde{c}) \in C([0,+\infty); H^2 \times H^2)$, differentiable a.e. in $t \in \mathbb{R}$. Let us assume that the initial condition $(\tilde{v}_0, \tilde{c}_0)$ is sufficiently close to a translate of the wave $(\bar{u}_x, \bar{c})(\cdot + \alpha_0)$ in the H^2 norm (and consequently, in the H^1 norm as well) such that for a certain $T > 0$ there exists an $\alpha \in C([0,T])$ with the properties of Lemma 6.1 (in particular, differentiable a.e. in $t \in [0,T]$). Then we can define a perturbation (v,c) by means of (128), which satisfies $(v,c)(\cdot,t) \in X_1 \subset H^1 \times H^1$ for each $t \geqslant 0$, and $(v,c) \in C([0,T]; H^2 \times H^2)$.

Make the change of variables $\xi := x + \theta t + \alpha(t)$, and substitute the decomposition (128) into system (135). The result is

$$\mu\dot{\alpha}(t)v_\xi + \mu\theta v_\xi + \mu v_t + v - s\int_{-\infty}^{\xi}\int_{-\infty}^{y} v(\zeta,t)\,d\zeta\,dy + \mu\dot{\alpha}(t)\bar{u}_{\xi\xi} + \mu\theta\bar{u}_{\xi\xi}$$

$$- s\int_{-\infty}^{\xi}\int_{-\infty}^{y} \bar{u}_\xi(\zeta,t)\,d\zeta\,dy + \bar{u}_\xi + \tau(0) - \tau\big(c(\xi,t) + \bar{c}(\xi)\big) = 0,$$

$$\dot{\alpha}(t)c_\xi + c_t + \theta c_\xi - Dc_{\xi\xi} - \epsilon v + \dot{\alpha}(t)\bar{c}_\xi + \theta\bar{c}_\xi - D\bar{c}_{\xi\xi} - \epsilon\bar{u}_\xi - R\big(c(\xi,t) + \bar{c}(\xi)\big) = 0.$$

From the equations for the wave (7) we find that for each $\xi \in \mathbb{R}$,

$$\mu\theta\bar{u}_{\xi\xi} + \bar{u}_\xi - \tau\big(\bar{c}(\xi)\big) - s\int_{-\infty}^{\xi}\int_{-\infty}^{y} \bar{u}_\xi(\zeta,t)\,d\zeta\,dy = C_0,$$

where $C_0 \in \mathbb{R}$ is a constant which we evaluate[4] by taking $\xi \to -\infty$. This yields $C_0 = -\tau(0)$.

Expand the nonlinear terms around the calcium front as

$$\tau(\bar{c} + c) - \tau(\bar{c}) = \tau'(\bar{c})c + N_1(\xi, c),$$
$$R(\bar{c} + c) - R(\bar{c}) = R'(\bar{c})c + N_2(\xi, c),$$

where

$$N_1(\xi, c) := \int_0^1 \tau''(\bar{c} + \zeta c)(1-\zeta)c^2\,d\zeta = \mathcal{O}\big(c^2\big),$$

$$N_2(\xi, c) := \int_0^1 R''(\bar{c} + \zeta c)(1-\zeta)c^2\,d\zeta = \mathcal{O}\big(c^2\big).$$

Then, substituting back the value of C_0 and using the wave equation for $\bar{c}$ in (7), we arrive at

$$\begin{aligned}\mu\dot{\alpha}(t)\bar{u}_{\xi\xi} + \mu\theta v_\xi + \mu v_t + v - s\mathcal{J}v - \tau'(\bar{c})c &= Q_1,\\ \dot{\alpha}(t)\bar{c}_\xi + c_t + \theta c_\xi - Dc_{\xi\xi} - \epsilon v - R'(\bar{c})c &= Q_2,\end{aligned} \tag{136}$$

[4] Notice that taking the limit $\xi \to +\infty$ we find the relation $\int_{\mathbb{R}} \bar{u}(\xi)\,d\xi = s^{-1}(\tau(0) - \tau(1))$.

where

$$
\begin{aligned}
Q_1 &:= N_1(\xi, c) - \mu\dot{\alpha}(t)v_\xi, \\
Q_2 &:= N_2(\xi, c) - \dot{\alpha}(t)c_\xi.
\end{aligned}
\tag{137}
$$

The resulting system (136) can be recast as

$$
\dot{\alpha}(t)(\bar{u}_{\xi\xi}, \bar{c}_\xi) + (v, c)_t = \mathcal{A}(v, c) + \mathbb{Q}, \tag{138}
$$

with

$$
\mathbb{Q} := \big(\mu^{-1}Q_1, Q_2\big)^\top. \tag{139}
$$

By local construction of α we have the constraints

$$
(v, c)(\cdot, t) \in X_1, \quad t \in [0, T], \tag{140}
$$

$$
(v, c)(\cdot, 0) \in X_1. \tag{141}
$$

The requirements (140) and $\mathcal{P}_1\mathbb{Q} \in X_1$ lead to the modulation equation for the phase, which prescribes the evolution of $\alpha(t)$. Taking the H^1-product of Eq. (138) with the adjoint eigenfunction $(\bar{\psi}, \bar{\phi})$ we get

$$
\Theta\dot{\alpha}(t) + \big\langle (v, c)_t, (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} = \big\langle \mathcal{A}(v, c), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} + \big\langle \big(\mu^{-1}Q_1, Q_2\big), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1}.
$$

Clearly,

$$
\big\langle \mathcal{A}(v, c), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} = \big\langle (v, c), \mathcal{A}^*(\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} = 0,
$$

and

$$
\big\langle (v, c)_t, (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} = \frac{d}{dt}\big\langle (v, c), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} = 0,
$$

because $(v, c) \in X_1$. Thus we arrive at

$$
\begin{aligned}
\Theta\dot{\alpha}(t) &= \big\langle \big(\mu^{-1}N_1, N_2\big), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} - \dot{\alpha}(t)\big\langle (v_\xi, c_\xi), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} \\
&= \big\langle \big(\mu^{-1}N_1, N_2\big), (\bar{\psi}, \bar{\phi})\big\rangle_{H^1\times H^1} + \dot{\alpha}(t)\big\langle (v, c), (\bar{\psi}_\xi, \bar{\phi}_\xi)\big\rangle_{H^1\times H^1},
\end{aligned}
$$

leading to the modulation equation

$$
\dot{\alpha}(t) = \frac{\Theta^{-1}\langle (\mu^{-1}N_1, N_2), (\bar{\psi}, \bar{\phi})\rangle_{H^1\times H^1}}{1 - \Theta^{-1}\langle (v, c), (\bar{\psi}_\xi, \bar{\phi}_\xi)\rangle_{H^1\times H^1}}. \tag{142}
$$

Whenever $\|(v, c)(\cdot, t)\|_{H^1\times H^1}$ is small, Eq. (142) prescribes the evolution behavior of $\alpha(t)$. We shall prove that the perturbation (v, c) remains small, due to the semigroup decay rates (123) in X_1.

Taking the projection of Eq. (138) onto X_1 we get

$$
(v, c)_t = \mathcal{A}_1(v, c) + \mathcal{P}_1\mathbb{Q}, \tag{143}
$$

with initial condition $(v_0, c_0) \in X_1$, given by $(v_0, c_0) = (\tilde{v}_0, \tilde{c}_0) - (\bar{u}_x, \bar{c})(\cdot + \alpha_0)$.

6.3. A priori estimates

The following estimates are obtained by using the semigroup estimates of Section 5.4. They imply that the decomposition (128) exists for all times, with (v,c) decaying in the H^1 norm.

Lemma 6.3. *There exists $\delta_2 > 0$ such that if the decomposition* (128) *exists for $t \in [0,T]$ and satisfies*

$$e^{\frac{1}{2}\omega_0 t}\big\|(v,c)(t)\big\|_{H^1\times H^1} \leqslant \delta_2, \quad \textit{for } 0 \leqslant t \leqslant T, \tag{144}$$

and if

$$\big\|(v_0,c_0)\big\|_{H^1\times H^1} \leqslant \eta, \tag{145}$$

for some $\eta > 0$, then

$$e^{\frac{1}{2}\omega_0 t}\big\|(v,c)(t)\big\|_{H^1\times H^1} \leqslant C\eta, \quad \textit{for } 0 \leqslant t \leqslant T. \tag{146}$$

Proof. Define

$$M_{\omega_0}(T) := \sup_{0\leqslant t\leqslant T}\big\{e^{\frac{1}{2}\omega_0 t}\big\|(v,c)(t)\big\|_{H^1\times H^1}\big\}.$$

Since the decomposition (128) exists for $t \in [0,T]$, we have that $\mathcal{G}[(\tilde{v},\tilde{c}),\alpha](t) = 0$ in $t \in [0,T]$, and $(v,c)(\cdot,t) \in X_1$. Thus (v,c) satisfies the projected system (143) with $(v_0,c_0) \in X_1$. Then the solution is determined by the variation of constants formula

$$(v,c)(\cdot,t) = \mathcal{S}_1(t)(v_0,c_0) + \int_0^t \mathcal{S}_1(t-\zeta)\mathcal{P}_1\mathbb{Q}\big(c(\zeta)\big)\,d\zeta, \quad t \in [0,T]. \tag{147}$$

From the exponential decay (123) of the semigroup $\mathcal{S}_1(t)$ we get

$$\big\|(v,c)(\cdot,t)\big\|_{H^1\times H^1} \leqslant Ce^{-\frac{1}{2}\omega_0 t}\big\|(v_0,c_0)\big\|_{H^1\times H^1} + C\int_0^t e^{-\frac{1}{2}\omega_0(t-\zeta)}\big\|\mathcal{P}_1\mathbb{Q}(\zeta)\big\|_{H^1\times H^1}\,d\zeta. \tag{148}$$

Notice that since $N_j = \mathcal{O}(c^2)$ we have

$$\mathbb{Q} = \mathcal{O}\big(c^2\big) + \mathcal{O}\big(\big|\dot{\alpha}(\zeta)\big|\big)(v_\xi, c_\xi).$$

From the modulated equation (142) we find the bound

$$\big|\dot{\alpha}(\zeta)\big| \leqslant \frac{1}{C_0-\delta_2}\mathcal{O}\big(c^2\big) \leqslant \mathcal{O}\big(c^2\big), \quad \zeta \in [0,T],$$

for $\delta_2 < C_0 = \mathcal{O}(|\Theta|)$ small. Therefore, using (144), we obtain

$$\|\mathcal{P}_1\mathbb{Q}\| \leqslant C\delta_2\big\|(v,c)(\zeta)\big\|_{H^1\times H^1},$$

for all $\zeta \in [0,T]$, and hence,

$$
\begin{aligned}
e^{\frac{1}{2}\omega_0 t}\left\|(v,c)(\cdot,t)\right\|_{H^1\times H^1} &\leqslant C\left\|(v_0,c_0)\right\|_{H^1\times H^1} + \bar{C}\delta_2\int_0^t \left\|(v,c)(\cdot,\zeta)\right\|_{H^1\times H^1} d\zeta \\
&\leqslant C\left\|(v_0,c_0)\right\|_{H^1\times H^1} + \tilde{C}\delta_2 M_{\omega_0}(T).
\end{aligned}
$$

Taking the supremum on the left-hand side for $t \in [0,T]$ we find that

$$
M_{\omega_0}(T)(1-\tilde{C}\delta_2) \leqslant C\left\|(v_0,c_0)\right\|_{H^1\times H^1}.
$$

We can thus take $\delta_2 > 0$ sufficiently small (in fact, $\delta_2 < \frac{1}{2}\min\{1/\tilde{C}, C_0\}$), and independent of $T > 0$ such that

$$
M_{\omega_0}(T) \leqslant C\left\|(v_0,c_0)\right\|_{H^1\times H^1} \leqslant C\eta,
$$

as claimed. □

6.4. Asymptotic behavior

Lemma 6.4. *There exists $\eta_0 > 0$ sufficiently small such that, for each $\alpha_0 \in \mathbb{R}$, if*

$$
\left\|(v_0,c_0)\right\|_{H^1\times H^1} < \eta_0, \tag{149}
$$

then the decomposition (128) *can be extended for all times and such that*

$$
\left\|(v,c)(\cdot,t)\right\|_{H^1\times H^1} \leqslant Ce^{-\frac{1}{2}\omega_0 t}\eta_0, \tag{150}
$$

for all $t \in [0,+\infty)$.

Proof. Let E be the set of all positive numbers $T > 0$ such that there exists a decomposition of the form (128) and such that estimate (144) holds. Let $T_* > 0$ be as in Lemma 6.2. Suppose that

$$
\left\|(\tilde{v}_0,\tilde{c}_0)(\cdot) - (\bar{u}_x,\bar{c})(\cdot+\alpha_0)\right\|_{H^1\times H^1} = \left\|(v_0,c_0)\right\|_{H^1\times H^1} < \eta_1,
$$

for some $\eta_1 > 0$. Then by continuity of the global solution $(\tilde{v},\tilde{c}) \in C([0,+\infty); H^1\times H^1)$ with initial condition $(\tilde{v}_0,\tilde{c}_0)$ (see Proposition A.2 and Remark A.3), given $\delta_0 > 0$ as in Lemma 6.1, there exists $\eta_1 > 0$ sufficiently small such that

$$
\sup_{0\leqslant t\leqslant T_1}\left\|(v,c)(\cdot,t)\right\|_{H^1\times H^1} = \sup_{0\leqslant t\leqslant T_1}\left\|(\tilde{v},\tilde{c})(\cdot,t) - (\bar{u}_x,\bar{c})(\cdot+\theta t+\alpha_0)\right\|_{H^1\times H^1} < \delta_0,
$$

for some $T_1 > 0$. Therefore, by Lemma 6.1 the decomposition exists for $t \in [0,T_1]$.

Choose $\eta_1 > 0$ sufficiently small such that the estimate (144) holds with $T = T_1$. This can be done because the mapping $t \mapsto \|(\tilde{v},\tilde{c})(\cdot,t)\|_{H^1\times H^1}$ is continuous. Therefore, there exists $T_1 \in E$ with $T_1 > 0$. This yields $0 < \sup E =: T_m \leqslant +\infty$.

Suppose that $T_m < +\infty$. Now we choose $\eta_0 > 0$ such that $C\eta_0 < \frac{1}{2}\min\{\delta_2,\delta_0,\eta_1\}$, where $C > 0$ is the constant of Lemma 6.3. Then for initial conditions satisfying (149) we have by Lemma 6.3 that

$$
e^{\frac{1}{2}\omega_0 t}\left\|(v,c)(\cdot,t)\right\|_{H^1\times H^1} \leqslant C\eta_0 < \delta_2,
$$

for all $t \in [0, T_m]$. Since $C\eta_0 < \delta_0$, by Lemma 6.2 the decomposition can be continued for $t \in [T_m, T_m + T_*]$ with $T_* > 0$. By continuity, estimate (144) remains valid in $[T_m, T_m + t]$ for $t < \varepsilon$ sufficiently small, contradicting the definition of T_m. Therefore $T_m = +\infty$, and the conclusion follows. $\square$

Lemma 6.4 implies that the decomposition (128) holds for all $t \in [0, +\infty)$ and therefore, that α satisfies the modulation equation (142) for all times. Since (150) holds for all $t \geqslant 0$ with $\eta_0 > 0$ small, we obtain

$$\begin{aligned}\left|\dot{\alpha}(t)\right| &= \frac{|\Theta^{-1}\langle(\mu^{-1}N_1, N_2), (\bar{\psi}, \bar{\phi})\rangle_{H^1\times H^1}|}{|1 - \Theta^{-1}\langle(v, c), (\bar{\psi}_\xi, \bar{\phi}_\xi)\rangle_{H^1\times H^1}|} \\ &\leqslant \frac{CC_1}{1 - C_2\eta_0}\left\|c(\cdot, t)\right\|_{H^1}^2 \leqslant \bar{C}\eta_0^2 e^{-\omega_0 t},\end{aligned}$$

as $N_j = \mathcal{O}(c^2)$. This shows that $|\dot{\alpha}(t)| \lesssim e^{-\omega_0 t} \to 0$ as $t \to +\infty$, and therefore

$$\alpha(t) = \alpha_0 + \int_0^t \dot{\alpha}(\zeta)\, d\zeta \to \alpha_\infty \in \mathbb{R},$$

as $t \to +\infty$. Moreover,

$$\left|\alpha(t) - \alpha_\infty\right| \leqslant C\eta_0 e^{-\omega_0 t}. \tag{151}$$

Finally, (150) and the (now) global decomposition (128) imply the desired asymptotic decay for the perturbation,

$$\left\|(\tilde{v}, \tilde{c})(\cdot, t) - (\bar{u}_x, \bar{c})\big(\cdot + \theta t + \alpha(t)\big)\right\|_{H^1\times H^1} \leqslant C\eta_0 e^{-\frac{1}{2}\omega_0 t} \to 0, \tag{152}$$

as $t \to +\infty$. Notice that since $\alpha(t)$ has a limit, we can write

$$\begin{aligned}&(\tilde{v}, \tilde{c})(x, t) - (\bar{u}_x, \bar{c})(x + \theta t + \alpha_\infty) \\ &\quad = (v, c)\big(x + \theta t + \alpha(t), t\big) + (\bar{u}_x, \bar{c})\big(x + \theta t + \alpha(t)\big) - (\bar{u}_x, \bar{c})(x + \theta t + \alpha_\infty) \\ &\quad = (v, c)\big(x + \theta t + \alpha(t), t\big) + (\bar{u}_{xx}, \bar{c}_x)\big(x + \theta t + \big(\alpha(t) - \alpha_\infty\big)\beta\big)\big(\alpha(t) - \alpha_\infty\big),\end{aligned}$$

for some $0 \leqslant \beta \leqslant 1$, which means that for $t > 0$ fixed, we have

$$\begin{aligned}&\left\|(\tilde{v}, \tilde{c})(\cdot, t) - (\bar{u}_x, \bar{c})(\cdot + \theta t + \alpha_\infty)\right\|_{H^1\times H^1} \\ &\quad \leqslant C\big(\left\|(v, c)(\cdot, t)\right\|_{H^1\times H^1} + \left\|(\bar{u}_{xx}, \bar{c}_x)(\cdot)\right\|_{H^1\times H^1}\left|\alpha(t) - \alpha_\infty\right|\big) \\ &\quad \leqslant \tilde{C}\eta_0 e^{-\omega_0 t},\end{aligned} \tag{153}$$

because of (151) and (150), proving the asymptotic orbital stability by letting $t \to +\infty$.

6.5. Proof of Theorem 3

By hypothesis, for $(\tilde{u}_0, \tilde{c}_0) \in H^2 \times H^1$ and $\alpha_0 \in \mathbb{R}$ satisfying (28), we then have initial conditions

$$(v_0, c_0)(\cdot) := (\tilde{u}_{0x}, \tilde{c}_0)(\cdot) - (\bar{u}_x, \bar{c})(\cdot + \alpha_0) \in H^1 \times H^1,$$

satisfying (149). Therefore there exists a decomposition of the form (128) for all times because of Lemma 6.4. Moreover, the solution lies in $(\tilde{v}, \tilde{c}) \in C([0, +\infty); H^1 \times H^1)$ (by Proposition A.2 and Remark A.3), with the perturbation satisfying (150). Moreover, there exists $\alpha_\infty = \lim \alpha(t)$ as $t \to +\infty$ such that $|\alpha_0 - \alpha_\infty| < C\eta_0$ because of (151) and with the asymptotic decay (153). This shows (29).

Moreover, if the initial condition is regular enough, $(\tilde{u}_0, \tilde{c}_0) \in (W^{1,1} \cap H^3) \times H^2$, then $(v_0, c_0) \in (L^1 \cap H^2) \times H^2$ and both the solution $(\tilde{v}, \tilde{c})$ and the perturbation (v, c) belong to $C([0, +\infty); H^2 \times H^2)$ by global existence of strong solutions, with same decay in $H^1 \times H^1$. Thus, the perturbation (v, c), which by construction lies in X_1, satisfies the nonlinear system (136). This implies that v is determined by the variation of constants formula (see (93)),

$$\hat{v}(k, t) = e^{-a(k)t} \hat{v}_0(k) + \int_0^t e^{-a(k)(t-\sigma)} \big(\big(\tau'(\bar{c})c\big)\widehat{\ }(k, \sigma) + \hat{Q}_1\big(c(\sigma)\big)\big)\, d\sigma,$$

with $(\hat{Q}_1, \hat{Q}_2)(c) := (\mathcal{P}_1 \mathbb{Q}(c))\widehat{\ }(k, t)$. Since $(v, c)_t \in L^1([0, +\infty); H^2 \times H^2)$, we can easily verify that the function $\hat{h}(k, t) := (\tau'(\bar{c})c)\widehat{\ }(k, t) + \hat{Q}_1(c(t))$ satisfies the hypothesis of Lemma 5.17. Thus, we conclude that we can define the antiderivative of v, namely $u \in C([0, +\infty); H^2)$ with $u(t) \in H^3$ for each $t > 0$, such that $u_x = v$ a.e., and therefore, there exists a solution $(\tilde{u}, \tilde{c}) \in C([0, +\infty); H^2 \times H^1)$ to the original system (1), with $(\tilde{u}, \tilde{c})(t) \in H^3 \times H^2$ for each $t > 0$ of the form (9).

By the same arguments as in the proof of Theorem 2, we can readily see that, $\|v(t)\|_{L^2}$ being uniformly bounded for all time t large, implies that $\|u(t)\|_{L^2} \leqslant C$ for all t large, with some $C > 0$. This yields an L^∞ estimate for the perturbation,

$$\big\|(u, c)(\cdot, t)\big\|_{L^\infty} \leqslant C_0 \big\|(u_x, c_x)(\cdot, t)\big\|_{L^2} \big\|(u, c)(\cdot, t)\big\|_{L^2} \leqslant C C_0 \eta_0 e^{-\omega_0 t},$$

as $t \to +\infty$. This shows (30). The result is now proved.

7. Discussion

7.1. General remarks

A key ingredient of the present nonlinear stability analysis is the establishment of suitable decay properties for the solutions to the linearized equations around the wave. For that purpose it is fundamental to perform a detailed spectral analysis. In this paper, this task is performed in a non-standard fashion by studying the associated first order formulation with the eigenvalue as a parameter [1]. This approach is suitable for our needs, as the equations have mixed partial derivatives. We emphasize that the spectral stability can be proved under this setting using quite general Evans function tools. The linear problem is then written in terms of the gradient of deformation u_x. Such a reformulation is not only more convenient from a physical viewpoint, but it has considerable technical advantages as well. For example, using the same methods employed in Section 5.2.1, it is possible to construct a solution operator $\Sigma(t) : L^2 \times L^2 \to L^2 \times L^2$ associated to the linear equations (13), acting on the original variables (u, c). This family of operators, however, does not necessarily constitute a strongly continuous semigroup. This observation is a consequence of the non-standard nature of the original problem. The reader might feel as well inclined to study the linear operators defined in (106) directly in order to show the existence of the decaying semigroup. Nevertheless, the standard semigroup theory seems to be hard to apply, because the algebraic curves bounding the essential spectrum suggest that the operator is not sectorial (see Remark 4.10), and the generalized Hille–Yosida theorem requires all powers in the resolvent estimates [9] to conclude existence. Moreover, the operator is clearly non-dissipative due to the variable coefficient term $R'(\bar{c})$ with alternate sign. Our constructive approach provides, in addition, a description of the dynamics of the solutions to the evolution equations taking place in a subset of double mean zero conditions for the deformation gradient, which are naturally induced by the elastic equation.

7.2. Comments on modeling

The model of Lane et al. [24] is based on the mechanochemical properties of the cytoskeleton material introduced by [27], in which the key elements are the deformation tensor u and the concentration c of free calcium. They assume that the egg is composed of a gel-like substance with elastic properties (deformable), and that there are certain molecules embedded in such material (called *acto-myosin* molecules) exerting active contraction stress and sensitive to free calcium concentration (hence the term $\tau(c)$). The chemical phenomenon is that of the free calcium on the egg's surface, which is released by autocatalytic self-estimulated process (the term $R(c)$), and which diffuses freely. Moreover, more calcium is seen as cytoplasm is stretched (stretch activation expressed by the term ϵu_x).

The first observation is that system of equations (1) comes from the simplifying assumption that the deformation u has only one component, in a one-dimensional strip. Lane et al. [24] consider this hypothesis based on phenomenological observations, namely, that the post-fertilization waves involve motion predominantly in the tangential direction, normal to the meridians [40]. Also, they make the observation that only one elastic component is not mathematically inconsistent because the material is assumed to be compressible. The numerical stability analysis presented in [24] deals with the one-dimensional version as well. The original model is, however, multidimensional. Considering a full three-dimensional elastic displacement $u \in \mathbb{R}^3$, the system of equations has the form

$$\begin{gathered} \operatorname{div}\big(\mu_1 E_t + \mu_2(\operatorname{div} u)_t I + \nu E + (1-\nu)(\operatorname{div} u)I - \tau(c)I\big) - su = 0, \\ c_t - D\Delta c - R(c) - \epsilon \operatorname{div} u = 0, \end{gathered} \tag{154}$$

where $E = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the strain tensor, μ_i are the bulk and shear viscosities, ν is the elastic modulus, and I is the identity matrix in $\mathbb{R}^{3\times 3}$. The stability of waves traveling in the tangential direction, but subject to multidimensional perturbations, is an interesting open problem (see discussion below).

Our second remark is that one of the main difficulties in the present stability study is the non-standard form of the equations, due to the presence of mixed derivatives in the equation for the elastic displacement. Notice that the second derivative in the spatial variables is not, in general, invertible. This problem could be avoided if there were a term of the form u_{tt}, for which the spectral analysis becomes the standard one for linearized operators. The response of the cytoplasm material proposed in [27,24] precludes inertial terms in the elastic force balance equation. That is why an inertial term of the form u_{tt} in the elastic equation in (1) has been neglected. Thus, one could regard system (1) as a non-inertial limit of a standard system, for which the stability analysis can be performed in the classical fashion. It would be interesting to elucidate whether the stability properties found in the present context come from a limiting behavior of a model of such kind.

7.3. Comments on multidimensional stability

We conjecture that the methodology and strategy of the present analysis can be carried out in the context of the multidimensional model (154), and that the results (at least at the spectral level) can be extrapolated to the case of transversal perturbations of the one-dimensional wave. If we consider the same traveling wave solutions in the tangential directions, but allowing perturbations in the azimuthal directions, we arrive at a spectral system which, recast as a first order system like (15), has the generic form

$$W_x = \mathbb{A}^\epsilon(x, \tilde{\xi}, \lambda)W$$

indexed again by parameter ϵ, but where $\tilde{\xi}$ represents the frequencies in the transversal azimuthal variables. The system has constant coefficients in the transversal components of the deformation (because the waves are assumed to travel tangentially), and therefore the whole spectral analysis of Section 4 could be carried out in a similar fashion. It is important to remark that the Evans function machinery for multidimensional problems is already well-developed (see [43–45,34]). The existence of the semigroup and exponential decay in the projected space could be studied following the blueprint of this paper. It is an open question whether nonlinear asymptotic stability holds under small multidimensional perturbations as well.

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Appendix A. Global existence of strong solutions

In this appendix we discuss the global existence of solutions to the nonlinear system of equations (23) with initial condition

$$(v_0, c_0) := (u_{0x}, c_0) \in H^2 \times H^2. \tag{155}$$

Without loss of generality we put $s = \mu = D = 1$. System (23) (or equivalently (135)) arises from recasting the original system of equations (1) in terms of the deformation gradient $v = u_x$. Recall that $\mathcal{J} : L^1 \cap H^2 \cap \mathcal{U} \subset H^2 \to H^2$ is the operator densely defined in (91) (see Corollary 5.9). System (135) defines a nonlinear evolution problem of the form

$$(v, c)_t = \mathcal{L}(v, c) + F(c), \tag{156}$$

where

$$\big(\mathcal{L}(v, c)\big)^\top := \begin{pmatrix} \mathcal{J} - 1 & 0 \\ \epsilon & \partial_x^2 \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix} \tag{157}$$

is a densely defined closed operator in $H^2 \times H^2$ with domain $\mathcal{D}(\mathcal{L}) = (L^1 \cap H^2 \cap \mathcal{U}) \times H^2$, and the nonlinear term is $F(c) := (\tau(c) - \tau(0), R(c))$.

Lemma A.1. *For all $0 \leqslant \epsilon < 1$, $\mathcal{L}$ generates a C_0-semigroup of contractions in $H^2 \times H^2$, which we denote as $\{e^{t\mathcal{L}}\}_{t \geqslant 0}$.*

Proof. The result follows from the classical Hille–Yosida theorem [9,32]. We only need to verify that the resolvent set $\rho(\mathcal{L})$ of $\mathcal{L}$ contains $\mathbb{R}_+$ and that for every $\lambda \in \mathbb{R}$, $\lambda > 0$, there holds the resolvent estimate $\|(\mathcal{L} - \lambda)^{-1}\| \leqslant \lambda^{-1}$.

Suppose $(\mathcal{L} - \lambda)(v, c) = (f, g)$ for some $(f, g) \in H^2 \times H^2$, $\lambda \in \mathbb{C}$. Equivalently,

$$\begin{aligned} &(1 + \lambda)v - \mathcal{J} v = f, \\ &\lambda c - c_{xx} - \epsilon v = g. \end{aligned} \tag{158}$$

Take the real part of the L^2-product of first equation with v and integrate by parts. The result is

$$(1 + \operatorname{Re}\lambda)\|v\|_{L^2}^2 - \operatorname{Re}\langle v, \mathcal{J} v\rangle_{L^2} = \operatorname{Re}\langle v, f\rangle_{L^2}.$$

Since $-\langle v, \mathcal{J} v\rangle_{L^2} \geqslant 0$ for all $v \in L^2$, we get

$$\|v\|_{L^2}^2 \leqslant (\operatorname{Re}\lambda + 1)^{-1}\|f\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\|f\|_{L^2}, \tag{159}$$

assuming $\operatorname{Re}\lambda > 0$. Proceed in the same fashion for the equation for c; this yields

$$(\operatorname{Re}\lambda)\|c\|_{L^2}^2 + \|c_x\|_{L^2}^2 = \epsilon \operatorname{Re}\langle c, v\rangle_{L^2} + \operatorname{Re}\langle c, g\rangle_{L^2} \leqslant \|c\|_{L^2}^2\big(\epsilon\|v\|_{L^2} + \|g\|_{L^2}\big),$$

or

$$\|c\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\big(\epsilon\|v\|_{L^2} + \|g\|_{L^2}\big).$$

Substitute (159) in last equation and assume $\epsilon < 1$, $\lambda \in \mathbb{R}$, $\lambda > 0$, to obtain, combining with (159), that

$$\big\|(v, c)\big\|_{L^2\times L^2} \leqslant \lambda^{-1}\big\|(f, g)\big\|_{L^2\times L^2}.$$

By differentiating the equation for v and taking the L^2-product with v_x we arrive at

$$\|v_x\|_{L^2}^2 \leqslant (\operatorname{Re}\lambda + 1)^{-1}\|f_x\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\|f_x\|_{L^2},$$

after noticing that $(\mathcal{J} v)_x = \mathcal{J} v_x$ and because of the sign of $-\langle v_x, \mathcal{J} v_x\rangle_{L^2} \geqslant 0$. Differentiating twice we arrive at the equation $(1+\lambda)v_{xx} - v = f_{xx}$; multiplying by v_{xx} and integrating by parts we get, in the same fashion,

$$\|v_{xx}\|_{L^2}^2 \leqslant (\operatorname{Re}\lambda + 1)^{-1}\|f_{xx}\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\|f_{xx}\|_{L^2}.$$

By differentiating the equation for c and by the same procedure as before we arrive at the estimates

$$\|c_x\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\big(\epsilon\|v_x\|_{L^2} + \|g_x\|_{L^2}\big),$$
$$\|c_{xx}\|_{L^2} \leqslant (\operatorname{Re}\lambda)^{-1}\big(\epsilon\|v_{xx}\|_{L^2} + \|g_{xx}\|_{L^2}\big).$$

Substitute the v_x estimate into the estimates for c_x as before and combine them together to arrive at

$$\big\|(v, c)\big\|_{H^2\times H^2} \leqslant \lambda^{-1}\big\|(f, g)\big\|_{H^2\times H^2}, \tag{160}$$

as claimed. Note $(\mathcal{L} - \lambda)^{-1}$ is invertible for all $\lambda \in \mathbb{R}_+$. The conclusion now follows. In particular notice that the semigroup is contractive,

$$\big\|e^{t\mathcal{L}}\big\| \leqslant 1, \quad t \geqslant 0. \qquad \square \tag{161}$$

We now state the main result of this section.

Proposition A.2 *(Global existence). Assume $\epsilon \in [0, 1)$. For every $(v_0, c_0) \in H^2 \times H^2$, the initial value problem* (156) *with initial condition $(v, c)(0) = (v_0, c_0)$ has a unique global strong solution $(v, c) \in$*

$C([0,+\infty); H^2 \times H^2)$, with (v,c) differentiable a.e. in $t \in (0,+\infty)$ and $(v,c)_t \in L^1((0,+\infty); H^2 \times H^2)$. The solution is determined by the variation of constants formula

$$(v,c)(t) = e^{t\mathcal{L}}(v_0,c_0) + \int_0^t e^{(t-\zeta)\mathcal{L}} F\big(c(\zeta)\big)\, d\zeta. \tag{162}$$

Proof. The result follows from standard semigroup theory [9,32]. For that purpose, notice that F does not depend on v nor on t, and that τ is smooth with compact support. Therefore, $\tau(c) - \tau(0)$ is uniformly Lipschitz as a mapping $H^2 \mapsto H^2$. Moreover, as $\bar{c} \in [0,1]$ and in view that we are interested in small perturbations $c \in H^2$, we can assume that $R(c)$ has an extension outside $[0,1]$ which is locally Lipschitz as a mapping $R(c) : H^2 \to H^2$. (It suffices to consider an extension in $C^{1,1/2}$ by the compact embedding $H^2(\Omega) \hookrightarrow C^{1,1/2}(\Omega)$, and taking Ω bounded with $[0,1] \subset \Omega$.)

Thus, without loss of generality, we can assume the nonlinear term $F(c)$ is locally Lipschitz in H^2. Let $L(K)$ be the Lipschitz constant of F, such that

$$\big\| F(c_1) - F(c_2) \big\|_{H^2} \leqslant L(K) \|c_1 - c_2\|_{H^2},$$

for all $\|c_1\|_{H^2} \leqslant K$, $\|c_2\|_{H^2} \leqslant K$.

By a direct application of Theorem 6.1.4 in [32, pp. 185–187], there exists $0 < T_{\max} \leqslant +\infty$ such that the initial value problem has a unique mild solution $(v,c) \in C([0,T_{\max}); H^2 \times H^2)$ determined by formula (162). Moreover, if $T_{\max} < +\infty$, then $\lim_{t \uparrow T_{\max}} \|(v,c)\|_{H^2 \times H^2} = +\infty$. Global existence ($T_{\max} = +\infty$) will follow if we can show that for every $T > 0$ there exists $M_T > 0$ such that $\sup_{t \in [0,T]} \|(v,c)\|_{H^2 \times H^2} \leqslant M_T$. Observe from formula (162) that it suffices to prove the bound for $\|c(t)\|_{H^2}$, since the semigroup is contractive and the nonlinear terms do not depend on v.

Define now $K_T := \sup_{t \in [0,T]} \|c(t)\|_{H^2}$. Then, by the local Lipschitz condition, $\|F(c(\zeta))\|_{H^2} \leqslant L(K_T)\|c(\zeta)\|_{H^2}$ for all $0 \leqslant \zeta \leqslant T$. Substituting in (162) and using contractiveness (161), we find that

$$\big\|c(t)\big\|_{H^2} \leqslant \big\|(v,c)(t)\big\|_{H^2} \leqslant \big\|(v_0,c_0)\big\|_{H^2 \times H^2} + C_T \int_0^t \big\|c(\zeta)\big\|_{H^2}\, d\zeta,$$

for all $t \in [0,T]$. By Gronwall's lemma,

$$\big\|c(t)\big\|_{H^2} \leqslant \big\|(v_0,c_0)\big\|_{H^2 \times H^2}\big(1 + TC_T e^{TC_T}\big) =: M_T \big\|(v_0,c_0)\big\|_{H^2 \times H^2},$$

for all $0 \leqslant t \leqslant T < T_{\max}$. Assuming $T_{\max} < +\infty$, last estimate shows that $\sup_{[0,T_{\max}]} \|c(t)\|_{H^2}$ (and hence $\sup_{[0,T_{\max}]} \|(v,c)(t)\|_{H^2 \times H^2}$ as well) is finite. This contradicts $T_{\max} < +\infty$ because of Theorem 6.1.4 in [32]. Therefore we conclude that $T_{\max} = +\infty$ and we have a global solution.

Finally, we observe that $H^2 \times H^2$ is a reflexive space, and that the nonlinear terms do not depend on $t \in \mathbb{R}_+$; therefore, by Theorem 6.1.6 in [32, p. 189], the solution is a strong solution in the sense that (v,c) is differentiable a.e. in $t \in (0,+\infty)$, with $(v,c)_t \in L^1((0,+\infty); H^2 \times H^2)$. □

Remark A.3. By the same arguments, and since $\mathcal{L}$ generates a C_0-semigroup of contractions in $H^1 \times H^1$ (a clear fact by inspection of the proof of Lemma A.1), we also have global existence of solutions $(v,c) \in C([0,+\infty); H^1 \times H^1)$, with $(v,c)_t \in L^1([0,+\infty); H^1 \times H^1)$ for initial conditions merely in $(v_0,c_0) \in H^1 \times H^1$.

Appendix B. Resolvent bounds: Proof of Proposition 5.31

In this section we prove uniform estimates (121) and (122), valid for large $\operatorname{Re}\lambda$ and for large $|\operatorname{Im}\lambda|$, respectively and independently, in the unstable half plane $\operatorname{Re}\lambda \geqslant 0$.

Let $(f,g) \in H^1 \times H^1$ and $(v,c) \in \hat{\mathcal{D}}$, the domain of $\mathcal{A}$ as defined in (108). For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$, consider the equations

$$\mu\theta v_x + (1+\mu\lambda)v - s\mathcal{J}v - \tau'(\bar{c})c = \mu f, \tag{163}$$

$$\lambda c + \theta c_x - Dc_{xx} - \epsilon v - R'(\bar{c})c = g. \tag{164}$$

This system of equations corresponds to a resolvent system of the form

$$(\lambda - \mathcal{A})(v,c) = (f,g). \tag{165}$$

First, we are going to prove estimate (122). Take the L^2-product of c with (164) and integrate by parts. The result is

$$\lambda\|c\|_{L^2} + \theta\langle c, c_x\rangle_{L^2} + D\|c_x\|_{L^2}^2 - \epsilon\langle c, v\rangle_{L^2} - \int_{\mathbb{R}} R'(\bar{c})|c|^2\,dx = \langle c, g\rangle_{L^2}. \tag{166}$$

Take the real part and apply Cauchy–Schwarz inequality to obtain the estimate

$$(\operatorname{Re}\lambda - M_1)\|c\|_{L^2} \leqslant \epsilon\|v\|_{L^2} + \|g\|_{L^2}, \tag{167}$$

where $M_1 = \max_{c\in[0,1]}|R'(c)| > 0$. Now take the L^2-product of v with (163) and integrate by parts to get

$$\mu\theta\langle v, v_x\rangle_{L^2} + (1+\mu\lambda)\|v\|_{L^2}^2 - s\langle v, \mathcal{J}v\rangle_{L^2} - \big\langle v, \tau'(\bar{c})c\big\rangle_{L^2} = \mu\langle v, f\rangle_{L^2}. \tag{168}$$

Note that since $v \in H^2 \cap L^1 \cap \mathcal{U}$, then we can find $V \in H^4$ such that $V = \mathcal{J}v$ and $V_{xx} = v$. Therefore $\langle v, \mathcal{J}v\rangle = -\|V_x\|_{L^2}^2 < 0$. Thus, taking the real part of (168) we get the estimate

$$(1+\mu\operatorname{Re}\lambda)\|v\|_{L^2} \leqslant M_2\|c\|_{L^2} + \mu\|f\|_{L^2}, \tag{169}$$

where $M_2 := \max_{c\in[0,1]}|\tau'(c)| > 0$.

At this point we make the observation that if we merely assume that λ has non-negative real part, then estimate (169) readily implies that

$$\|v\|_{L^2} \leqslant M_2\|c\|_{L^2} + \mu\|f\|_{L^2}. \tag{170}$$

Now we are going to assume that $\operatorname{Re}\lambda$ is large enough. Let

$$\operatorname{Re}\lambda \geqslant \hat{K}_0 := \max\{2M_1, M_1+1\} > 0.$$

This implies that

$$0 < (\operatorname{Re}\lambda - M_1)^{-1} \leqslant 1, \tag{171}$$

$$0 < (\operatorname{Re}\lambda - M_1)^{-1} \leqslant 2(\operatorname{Re}\lambda)^{-1}, \tag{172}$$

$$0 < (1+\mu\operatorname{Re}\lambda)^{-1} \leqslant \frac{2}{\mu}(\operatorname{Re}\lambda)^{-1}. \tag{173}$$

Hence, substitution of (167) into (169) yields

$$\big(1+\mu\operatorname{Re}\lambda - M_2\epsilon(\operatorname{Re}\lambda - M_1)^{-1}\big)\|v\|_{L^2} \leqslant M_2(\operatorname{Re}\lambda - M_1)^{-1}\|g\|_{L^2} + \mu\|f\|_{L^2}. \tag{174}$$

In view of $M_2 > 0$, choose ϵ_0 such that $0 < \epsilon_0 < \frac{1}{2}M_2^{-1}$; whence, for each $\epsilon \in [0, \epsilon_0)$ we have $M_2\epsilon(\operatorname{Re}\lambda - M_1)^{-1} \leqslant M_2\epsilon < \frac{1}{2} < \frac{1}{2}(1 + \mu \operatorname{Re}\lambda)$, leading to the estimate

$$\|v\|_{L^2} \leqslant \frac{2M_2\|g\|_{L^2}}{(1 + \mu \operatorname{Re}\lambda)(\operatorname{Re}\lambda - M_1)} + \frac{2\mu\|f\|_{L^2}}{1 + \mu \operatorname{Re}\lambda}.$$

Use (173) and (171) to arrive at

$$\|v\|_{L^2} \leqslant C(\operatorname{Re}\lambda)^{-1}\big(\|f\|_{L^2} + \|g\|_{L^2}\big), \tag{175}$$

for some uniform $C > 0$. Substitute (175) into (167) and use (171), (172) to get

$$\|c\|_{L^2} \leqslant C(\operatorname{Re}\lambda)^{-1}\big(\|f\|_{L^2} + \|g\|_{L^2}\big), \tag{176}$$

for some $C > 0$. Thus we have proved the uniform bound

$$\big\|(v, c)\big\|_{L^2\times L^2} \leqslant C(\operatorname{Re}\lambda)^{-1}\big\|(f, g)\big\|_{L^2\times L^2}. \tag{177}$$

In order to obtain the estimate for the derivatives, differentiate Eqs. (163) and (164); this yields the system

$$\mu\theta v_{xx} + (1 + \mu\lambda)v_x - s(\mathcal{J}v)_x - \tau'(\bar{c})c_x - \tau''(\bar{c})\bar{c}_x c = \mu f_x, \tag{178}$$

$$\lambda c_x + \theta c_{xx} - Dc_{xxx} - \epsilon v_x - R'(\bar{c})c_x - R''(\bar{c})\bar{c}_x c = g_x. \tag{179}$$

Take the L^2-product of c_x with (179), integrate by parts and take its real part to readily obtain the estimate

$$(\operatorname{Re}\lambda - M_1)\|c_x\|_{L^2} \leqslant \epsilon\|v_x\|_{L^2} + M_3\|c\|_{L^2} + \|g_x\|_{L^2}, \tag{180}$$

where $M_1 = \max|R'| > 0$ as before, and $M_3 := \sup_{x\in\mathbb{R}}|R''(\bar{c})\bar{c}_x| \geqslant 0$.

Observing that $\langle v_x, (\mathcal{J}v)_x\rangle_{L^2} = -\|v\|^2_{L^2}$, take the L^2-product of v_x with (178), integrate by parts and take the real part to arrive at

$$(1 + \mu \operatorname{Re}\lambda)\|v_x\|_{L^2} \leqslant M_4\|c\|_{L^2} + M_2\|c_x\|_{L^2} + \mu\|f_x\|_{L^2}, \tag{181}$$

where $M_2 = \max|\tau'| > 0$ as before, and with $M_4 := \sup_{x\in\mathbb{R}}|\tau''(\bar{c})\bar{c}_x| \geqslant 0$.

Substitute (180) into (181) and by a similar argument as before, we get that for each $0 \leqslant \epsilon \leqslant \epsilon_0 < \frac{1}{2}M_2^{-1}$ small and for $\operatorname{Re}\lambda \geqslant \hat{K}_0$ there holds the estimate

$$\|v_x\|_{L^2} \leqslant \tilde{C}_1(\operatorname{Re}\lambda)^{-1} + \tilde{C}_2(\operatorname{Re}\lambda)^{-1}\big(\|g_x\|_{L^2} + \|f_x\|_{L^2}\big), \tag{182}$$

for some uniform $\tilde{C}_i > 0$. Finally, take

$$K_2 := \max\{\hat{K}_0, 1\},$$

so that for each $\operatorname{Re}\lambda > K_2$, $(\operatorname{Re}\lambda)^{-1} \leqslant 1$ and previous estimates hold. Substitute (176) into (182) to obtain

$$\|v_x\|_{L^2} \leqslant C(\operatorname{Re}\lambda)^{-1}\big\|(f, g)\big\|_{H^1\times H^1}, \tag{183}$$

for some $C > 0$, all $\operatorname{Re}\lambda > K_2$. Use (171) and (172), together with (183) and (176) to obtain from (180) the estimate

$$\|c_x\|_{L^2} \leqslant C(\operatorname{Re}\lambda)^{-1}\big\|(f,g)\big\|_{H^1\times H^1}. \tag{184}$$

Combine bounds (184), (183) with (177) to conclude that there exist uniform constants $C_2 > 0$, $K_2 > 0$ such that

$$\big\|(v,c)\big\|_{H^1\times H^1} \leqslant C_2(\operatorname{Re}\lambda)^{-1}\big\|(f,g)\big\|_{H^1\times H^1}, \tag{185}$$

for all $\operatorname{Re}\lambda \geqslant K_2$ and provided that $\epsilon \in [0,\epsilon_0)$ is sufficiently small. This proves estimate (122) and part (ii) of Proposition 5.31.

Next, we prove estimate (121). Here we only assume that $\epsilon \geqslant 0$ and $\operatorname{Re}\lambda \geqslant 0$.

The L^2-product of c with Eq. (164) yields (166). Take its imaginary part to get the estimate

$$|\operatorname{Im}\lambda|\|c\|_{L^2} \leqslant \theta\|c_x\|_{L^2} + \epsilon\|v\|_{L^2} + \|g\|_{L^2}. \tag{186}$$

Since $\operatorname{Re}\lambda \geqslant 0$, taking the real part of (166) yields

$$\|c_x\|_{L^2}^2 \leqslant \hat{C}_1\big(\|c\|_{L^2}^2 + \|v\|_{L^2}^2 + \|g\|_{L^2}^2\big), \tag{187}$$

with some $\hat{C}_1 > 0$. We have observed that assuming $\operatorname{Re}\lambda \geqslant 0$, then estimate (170) holds. Hence, substitute (170) and (187) into (186) to obtain

$$\begin{aligned}|\operatorname{Im}\lambda|^2\|c\|_{L^2}^2 &\leqslant \hat{C}_2\big(\|c_x\|_{L^2}^2 + \|v\|_{L^2}^2 + \|g\|_{L^2}^2\big)\\ &\leqslant \hat{C}_3\|c\|_{L^2}^2 + \hat{C}_3\big(\|f\|_{L^2}^2 + \|g\|_{L^2}^2\big),\end{aligned}$$

for some uniform constant $\hat{C}_3 > 0$. Taking $|\operatorname{Im}\lambda| \geqslant K_1 := \sqrt{2\hat{C}_3} > 0$ there holds

$$\|c\|_{L^2}^2 \leqslant \|f\|_{L^2}^2 + \|g\|_{L^2}^2,$$

which combined with (187) and (170) leads to

$$\|c_x\|_{L^2}^2 \leqslant \hat{C}_4\big(\|f\|_{L^2}^2 + \|g\|_{L^2}^2\big).$$

Last bounds merge into

$$\|c\|_{H^1} \leqslant C\big\|(f,g)\big\|_{L^2\times L^2}, \tag{188}$$

for some uniform $C > 0$ and for $|\operatorname{Im}\lambda| \geqslant K_1$ large enough.

To arrive at the bounds for v, take the imaginary part of (168) to get

$$|\operatorname{Im}\lambda|\|v\|_{L^2} \leqslant \theta\|v_x\|_{L^2} + \|f\|_{L^2} + M_2\mu^{-1}\|c\|_{L^2}, \tag{189}$$

in view of $\langle v, \mathcal{J}v\rangle = -\|V_x\|_{L^2}^2 < 0$ and with $M_2 > 0$ defined as before. Once again, observe that $\langle v_{xx}, \mathcal{J}v\rangle = \|v\|_{L^2}^2$ after integration by parts. Thus, take the L^2-product of v_{xx} with Eq. (163), integrate by parts and take its real part to obtain

$$\begin{aligned}
&(1 + \mu \operatorname{Re}\lambda) \|v_x\|_{L^2}^2 + s\|v\|_{L^2}^2 \\
&\quad = \operatorname{Re}\big\langle v_x, \tau'(\bar{c})c_x\big\rangle_{L^2} + \operatorname{Re}\big\langle v_x, \tau''(\bar{c})\bar{c}_x c\big\rangle_{L^2} + \mu \operatorname{Re}\langle v_x, f_x\rangle_{L^2}.
\end{aligned}$$

Take $M_4 \geqslant 0$ and $M_2 > 0$ as before. Then last equation yields the bound

$$\|v_x\|_{L^2} \leqslant M_2\|c_x\|_{L^2} + M_4\|c\|_{L^2} + \mu\|f_x\|_{L^2}. \tag{190}$$

If $|\operatorname{Im}\lambda| \geqslant K_1$ we may use (188) to obtain

$$\|v_x\|_{L^2} \leqslant C\big(\big\|(f, g)\big\|_{L^2\times L^2} + \|f_x\|_{L^2}\big). \tag{191}$$

Combine last equation and (188) with (189). The result is

$$K_1\|v\|_{L^2} \leqslant |\operatorname{Im}\lambda|\|v\|_{L^2} \leqslant \hat{C}\big(\big\|(f, g)\big\|_{L^2\times L^2} + \|f_x\|_{L^2}\big),$$

for some uniform $\hat{C} > 0$. This estimate and (191) imply

$$\|v\|_{H^1} \leqslant C\big\|(f, g)\big\|_{H^1\times H^1}. \tag{192}$$

Combining (192) and (188) we conclude that there exist uniform constants $K_1 > 0$, $C_1 > 0$ such that

$$\big\|(v, c)\big\|_{H^1\times H^1} \leqslant C_1\big\|(f, g)\big\|_{H^1\times H^1}, \tag{193}$$

provided that $|\operatorname{Im}\lambda| \geqslant K_1$ and $\operatorname{Re}\lambda \geqslant 0$. Estimate (193) holds for all $\epsilon \geqslant 0$. This shows the resolvent bound (121) and the proposition is now proved.

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